Inflation Implications of Rising Government Debt*

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Abstract. The goverrnment's intertemporal budget constraint implies a relationship between the ratio of current liabilities to the primary deficit and future values for the deficit, narrow money, inflation, interest rates and GDP growth. We evaluate the ability of this framework to explain the fiscal behaviour of the G7 since 1970. We show how debt is normally financed through changes in the primary deficit (90%) with less substantial roles being played by inflation (2%) and GDP growth (5-10%). We then use this framework to consider the implications of demographic factors for government finances. Using projections for each countries future deficits and the impact on interest rates and growth rates we provide upper bounds on the impact of demography on inflation on the basis of unchanged fiscal policies and calculate the required fiscal adjustment necessary to maintain stable inflation.

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1. Introduction

Figures 1 and 2 show recent fiscal trends for 6 large industrialised nations. Levels of government indebtedness increased markedly during the 1970s but stabilised and then improved during the 1980s and 1990s only to show recently signs of further deterioration. With OECD countries experiencing an ageing population it is widely expected that government's fiscal positions will worsen yet further in coming decades - see Figure 3 for projections. These considerations raise three important issues i) is current fiscal policy sustainable? ii) how have OECD governments financed their fiscal deficits in recent decades? iii) What are the implications for inflation of these rising fiscal deficits? This paper seeks to provide insights to each of these three questuons.

Key to our analysis is the government's intertemporal budget constraint. In assessing the first question (on sustainability) we use the methodology of Giannitsarou and Scott (2006) and derive a log linear approximation to the intertemporal budget constraint. Using this framework we show how debt sustainability requires an equilibrium relationship between the market value of government debt, the stock of narrow money and the level of government revenue and expenditure. We show how to estimate this relationship and derive a measure of sustainability for 6 OECD countries and use this measure to characterise the dynamics of fiscal adjustment amongst these countries. We can also use this framework to consider how governments have historically financed their deficits. We show analytically how deviations from the equilibrium relationship between debt, money and the primary deficit have to be met through future changes in either primary deficits, money creation, real interest rates, inflation or GDP growth. Using the VAR methodology proposed by Campbell and Shiller (1988)

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we assess for the period 1960-2005 the relative contribution of each channel to financing fiscal activity. Our third and final focus is to use this framework to assess whether the substantial expected increase in fiscal deficits threatens current low levels of inflation. The link between fiscal deficits and inflation is much analysed theoretically (see *inter alia* Sargent and Wallace (1981), McCallum (1984), Leeper (1991), Sims (1994) and Woodford (1995). By contrast our focus is empirical - to determine the extent to which variations in inflation have helped maintain fiscal solvency and the role of fiscal imbalances in predicting future inflation. Our intention in so doing is not toascertain the specific channels through which fiscal policy might affect inflation or to deliver a verdict on the empirical relevance of the fiscal theory of the price level. Our intention is purely applied - to gauge how important the quantitive link between higher debt and higher inflation has been in the recent past and to draw implications for the future.

The plan of the paper is as follows. Section 2 introduces our particular approach to the intertemporal budget constraint. Section 3 estimates our key measure of fiscal sustainability - a ratio between government liabilities and the current primary deficit - and analyses the dynamics of fiscal adjustment. Section 4 estimates historically how governments have financed their activities during the twentieth century, offering a decomposition between deficits, seigniorage, interest rate fluctuations, inflation and GDP growth. Section 5 assesses the empirical relevance of one of the implications of our intertemporal budget constraint - that a measure of fiscal liabilities to current deficits should have predictive ability for future inflation. Section 6 then uses some projections of future fiscal deficits on the basis of unchanged fiscal policies to try and derive some implications for inflation of the demographic induced deterioration in public finances. A final section concludes.

2. Government Budget Constraint

The focus of our analysis is on how fiscal sustainability is achieved by governments. In particular we focus on the behaviour of the US, Japan, Germany, UK, Italy and Canada over the period 1960-2005 and the stability, or otherwise, of the market value of government debt. Therefore at the heart of our paper is the government budget constraint. Let G_t^* denote the nominal value of government expenditure, T_t^* the nominal value of government revenue, Y_t^* nominal GDP and B_t^* nominal debt and let $(1+\gamma_t)$ denote the growth in nominal GDP and ϒt the one year holding return on nominal government bonds $(1+r_t)$ Then we have

$$\Delta B_t = (G_t - T_t) + r_t B_{t-1} - B_{t-1}\frac{\gamma_t}{1+\gamma_t} \tag{1}$$

where $B_t = B_t^*/Y_t^*$, $G_t = G_t^*/Y_t^*$ and $T_t = T_t^*/Y_t^*$. In other words, the debt/GDP ratio increases by the ratio of the primary deficit to GDP ratio and is reduced by a nominal growth dividend $B_{t-1}\frac{\gamma_t}{1+\gamma_t}$ Our use of the one year holding return (e.g including both coupon payments and capital gains) means that our budget constraint is specified in terms of the market value of government debt rather than the stock of outstanding debt. This choice of market value data is motivated by Marcet and Scott (2005) who show, in the case of complete markets, the potential importance of variations in r_t offsetting fluctuations in the primary deficit and achieving debt stability.

Table 1 shows sample averages of the variables in (1) for each country[1]. The results show that only the US and UK achieved relatively stable debt - the other nations all saw sustained increases

[1] Full details of the data are contained in the Data Appendix.

in their debt/GDP ratio. Every country ran an average total deficit across the period with only the UK and Canada achieving an average primary surplus. However an important reason for debt not growing faster was the nominal growth dividend, and within this effect inflation plays a majority role. Examination of Table 1 suggests that concern that rising demographic deficits will lead to higher inflation may be justified, given the importance of inflation effects through the nominal growth dividend.

Equation (1) is a helpful organising framework but has a number of limitations. In particular it is backward looking, focusing on average movements. In order to take a forward looking approach and focus on how fiscal policy responds to shocks we need to utilise the intertemporal budget constraint. Giannitsarou and Scott (2006) show how the log linearisation approach to intertemporal budget constraints can be applied to the government. This approach has been widely used across a range of applications (e.g Campbell and Shiller (1987) and (1988) apply this approach to equity prices and dividends, Campbell and Shiller (1991) use it to analyse the yield curve; Lettau and Ludvigson (2001) examine the consumption -wealth ratio and its ability to predict capital gains; Gourinchas and Rey (2005) apply the framework to the balance of payments and derive a measure of external imabalance and consider its implications for forecasting exchange rates). A key difference that has to be dealt with when applying this approach to public finances is that both government expenditures and revenues seem to contain unit roots even when measured relative to GDP.

Consider the following nominal government budget constraint

$$G_t - T_t = B_t - \frac{\Upsilon_{t-1}}{\Pi_t Q_t} B_{t-1} + H_t - \frac{1}{\Pi_t Q_t} H_{t-1}.$$

where H_t denotes the ratio of monetary liabilities to GDP, Π_t the inflation rate $(1 + \pi_t)$, and Q_t is the growth in nominal GDP e.g $(1 + \gamma_t)$. To proceed further we need to make the following assumptions (representing minor modifications to standard assumptions made in the literature):

Assumption 1 : There exists a variable W_t such that $G_t/W_t, T_t/W_t, B_t/W_t$ and H_t/W_t are stationary.

Assumption 2 : The real and nominal interest rate, the growth rate of GDP, inflation and the growth rate of W_t (W_t/W_{t-1}) are stationary, with steady states i, r, γ, π and ω respectively.

Assumption 3 The No-Ponzi condition holds e.g

$$\lim_{N\to\infty} \left(\frac{1}{\mu_b}\right)^N (B_{t+N-1} + H_{t+N-1}) = 0$$

where μ_b denotes the growth adjusted real interest rate less growth in W_t This latter condition will hold if

$$(1 + \gamma)(1 + \omega) < 1 + r$$

Normally Assumption 1 is assumed by setting W_t=Y_t where Y_t is GDP but in the case of government expenditure and tax revenues this is not sufficient to transform the variables to stationarity - see the Appendix. However, as the derivation in the Appendix shows we do not need to formally identify the variable W_t in order to implement our approach, we merely need to assume that such a variable exists. W_t is a shared trend and can be interpreted either as a purely statistical term or as representing a particular economic variable. For instance, W_t could denote the stock of public assets. Data on this series is unavailable so cannot be included in our empirical analysis but our log linear approximation merely requires that such a variable exists rather than it has to be accurately measured[2].

Assumption 2 makes specific stationarity assumptions on our real and nominal interest rates and GDP growth and inflation. A comprehensive summary of unit root tests is given in the Appendix. As always the results of this tests are not everywhere uniform but they suggest that $G_t/Y_t, T_t/Y_t, H_t/Y_t$ are non-stationary, B_t/Y_t probably is non-stationary[3] and that inflation, nominal and real interest rates and GDP growth are all stationary. The latter results suggest that Assumption 2 is valid for our sample, with the exception of the assumed but untestable assumption that the first difference of W_t is stationary.

Under Assumptions 1-3 it can be shown (see Appendix) that

$$l_t \equiv (1-\frac{1}{\mu_b})b_{t-1} + \phi\frac{\mu_h}{\mu_b}(1-\frac{1}{\mu_h})h_{t-1} + \frac{1}{\mu_b}[(1-\mu_b)+(1-\mu_h)\phi]d_t = \tag{2}$$

$$= -\frac{1}{\mu_b}[(1-\mu_b)+(1-\mu_h)\phi]E_t\sum_{j=1}^{\infty}\left(\frac{1}{\mu_b}\right)^j \Delta d_{t+j}$$

$$+\frac{\phi}{\mu_b}(1-\frac{\mu_h}{\mu_b})E_t\sum_{j=0}^{\infty}\left(\frac{1}{\mu_b}\right)^j \Delta h_{t+j}$$

$$+\left(1-\frac{1}{\mu_b}\right)E_t\sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j\left\{-r_{t+j-1}+\phi\frac{\mu_h}{\mu_b}\pi_{t+j}+\left(1+\phi\frac{\mu_h}{\mu_b}\right)\gamma_{t+j}\right\}$$

where $x_t = \ln x_t$ e.g $b_{t-1} = \log B^*_{t-1}/Y^*_{t-1}$and $d_t =\lambda_g g_t + \lambda_\tau \tau_t$ where $\lambda_g = \frac{\bar{G}}{\bar{G}-\bar{T}}$, $\lambda_\tau = -\frac{\bar{T}}{\bar{G}-\bar{T}}$ and $\bar{G}$ and $\bar{T}$ denote steady state values of $(G_t/Y_t)/W_t)$ and $(T_t/Y_t)/W_t)$ respectively and $\mu_b > 1, \mu_h < 1$. The variable d_t is essentially a transformed version of the primary deficit which depends on the coefficients λ_g and λ_τ If $\lambda_g > 0$ and $\lambda_\tau < 0$ then l_t defines a relationship between government liabilities and the primary deficit, if $\lambda_g < 0$ and $\lambda_\tau > 0$ then instead the relationship is between government liabilities and the primary surplus. Because by assumption $\mu_b > 1, \mu_h < 1$ the coefficient on debt is positive and on money is negative regardless of the sign of .λ_g and λ_τ.

The left hand side of (2) pins down a long run equilibrium relationship between the market value of government debt, monetary liabilities and a version of the current primary deficit, d_t. The interpretation of this equation is that, for given values of mean interest rates and money holdings, there has to be a steady state relationship between debt and the primary deficit if debt is to be sustainable and for the intertemporal budget constraint to hold. Under our Assumption 2 (and given our empirical evidence on unit roots) the right hand side of (2) is stationary and we have that $E_t l_{t+j} = \kappa$, as $j \to \infty$ so that $l_t - \kappa$ is a natural measure of required fiscal adjustment if the

[2]Making W_t unobservable does however change the way we implement our approach. In particular it means that we have to estimate μ_b and μ_h from the data rather than calibrate them from sample period averages, as in Gourinchas and Rey (2005).

[3]Giannitsarou and Scott (2006) using data between 1700-2005 suggest that debt/GDP ratio is stationary over this longer period.

intertemporal budget constraint is to hold. When $l_t = 0$ then this equilibrium relationship is holding but for $l_t > 0$ debt is too high relative to current fiscal deficits. Further insights into (2) and l_t can be gained from considering the results of Trehan and Walsh (1988) and (1991) and Bohn (2004). The former show that the intertemporal budget constraint requires that the primary deficit and debt satisfy a cointegrating relationship. Given Assumption 2 and the fact that $E_t l_{t+j} = \kappa$ so too does our approximation to the intertemporal budget constraint[4]. Bohn (2004) focuses on an alternative insight - that debt sustainability requires a feedback rule from debt to deficits. Given Assumption 2 we know that l_t must be stationary which can be achieved through a feedback rule from debt to the deficit, although in this case the feedback is from a weighted average of marketable debt and monetary liabilities.

If the left hand side of (2) measures the degree of fiscal adjustment required then the right hand side of (2) tells us how this fiscal adjustment $(l_t - \kappa)$ is achieved. Fiscal adjustment can be achieved through either i) future improvements in the fiscal deficit $-\Delta d_{t+j}$ ii) issuing more monetary liabilities or iii) variations in the growth adjusted real interest rate $(r_{t+j} - \pi_{t+j} - \gamma_{t+j})$.The coefficients on each of the components of the growth adjusted real interest rate differ as the nominal dividend effect $(\pi_{t+j} - \gamma_{t+j})$ operates on both bonds and money while r_{t+j} affects only bonds. Equation (2) tells us that, if the intertemporal budget constraint holds, any deviations in the long run relationship between debt and deficits must help predict movements in either future primary deficits, money creation, nominal interest rates, inflation or GDP growth.

Using long run historical data (1700-2005 for the UK, 1870 to 2005 for the US) Giannitsarou and Scott (2006) consider the full implications of (2) for a wide range of macroeconomic issues - (how do governments finance their debt, have financing methods remained stable over time and in particular are they different during war time, does our long run equilibrium component help predict real interest rates and the term structure (crowding out), GDP growth, inflation and future deficits and what implications does this equation have for fiscal rules and limits on debt). Our focus in this paper is however more concentrated - we look at the behaviour of fiscal policy in 6 advanced nations and concentrate only on the inflationary implications of rising government debt.

3. Equilibrium Fiscal Policy

The left hand side of (2) defines our key equilibrium component - a long run relationship between debt, money and the primary deficit. In order to consider empirically how fiscal adjustment is achieved we need to estimate (2) and construct a measure of l_t. Because our key variables (b_t, h_t, g_t and τ_t- logs of $B_t^*/Y_t^*, H_t^*/Y_t^*, G_t^*/Y_t^*, T_t^*/Y_t^*$) all display evidence of non-stationary we have to estimate a cointegrating vector between these variables if we are to construct an estimate of l_t. We do so using Stock and Watson's (1993) Dynamic OLS estimator and estimate the following relationship.

$$g_t = \beta_1 \tau_t + \beta_2 b_{t-1} + \beta_3 h_{t-1} + \sum_{i=-k}^{k} (c_{i\tau} \Delta \tau_{t-i} + c_{ib} \Delta b_{t-i} + c_{ih} \Delta h_{t-i}) \tag{3}$$

where our model implies the restriction $\beta_2 + \beta_3 = 1$ and $\beta_1 = -\frac{\lambda_\tau}{1-\lambda_\tau}$.Table 2 shows the results from estimating this equation and the implied coefficient estimates for the parameters of interest. In each case we can accept the restriction that $\beta_2 + \beta_3 = 1$, normally at the 5% level. Equation (3)

[4]Strictly speaking (2) states that if the components of l_t are of order of integration N e.g $I(N)$ then the RHS is $I(N-1)$.

contains three parameters of interest for our model - $\beta_1, \beta_2, \beta_3$ - and we need to form estimates of $\phi, \mu_b, \mu_h, \lambda_t$. Further, with $\beta_2 + \beta_3 = 1$ we in effect have only two estimated parameters. In order to identify our parameters we need to utilise the fact that ϕ is the sample average of narrow money to government debt ratio $(H/B)$and μ_b=$\mu_h(1+\gamma)$.Using these sample means we can then construct our estimate of $l_t = (1 - \frac{1}{\mu_b})b_{t-1} + \phi\frac{\mu_h}{\mu_b}(1 - \frac{1}{\mu_h})h_{t-1} + \frac{1}{\mu_b}[(1-\mu_b) + (1-\mu_h)\phi]d_t$ where l_t denotes the deviation of the ratio of government liabilities from a measure of the primary deficit. The larger is l_t the larger is debt relative to the steady state level of deficits. Note that given the estimates of Table 2 the weights on government expenditure and tax revenue are of opposite sign and approximately equal in absolute value so that d_t is only mildy different from the primary deficit (and for Japan and the UK there is very little difference).

Figures 3a-f show our estimates of a transformed version of l_t for each country. The transformation is performed in order to improve the ease of interpretability. We construct $l_t^* = \text{-}l_t/\frac{\lambda_t}{\mu_b}[(1-\mu_b) + (1-\mu_h)\phi]$ so that l_t^* is the change in $\log(T_t^*/Y_t^*)$ needed to bring about fiscal equilibrium ($l_t = 0$) i.e we divide l_t through by the coefficient on taxes. Figure 3 converts this into an estimate of the required increase in T_t^*/Y_t^* at each point in time given the current period tax revenue/GDP ratio. Our estimates suggest that the discrepancy between debt and deficit in the US is currently on a par with the Reagan years, although it has shown some signs of improvement over the last year. After a protracted correction during the 1980s and 1990s the estimates suggest that Canadian public finances are now in rough balance while there is some evidence that German policy may be too tight. As is to be expected Japan's situation has shown a dramatic recent deterioration and that UK policy has seen a recent sharp deterioration in the ratio of debt to deficit. Finally after many consecutive years of fiscal improvements our estimates suggest that from a long term perspective Italian public finances have recently deteriorated significantly.

Our interpretation of l_t as a measure of deviations in the debt/deficit ratio from its steady state relationships means that it represents an alternative approach to measuring fiscal sustainability (see Blanchard et al (1989) and Polito and Wickens (2005) for alternative measures). Table 3 reports some summary statistics regarding fiscal adjustment for our coutnries during this period. The degree of imbalance varies between around plus and minus 5% for all economies, except for the US where the range is narrower (plus or minus 3%). In all cases fiscal adjustment is a highly persistent process although not a unit root process - as discussed above it is critical for our analysis that l_t is stationary. Fiscal adjustment is a protracted process, with a half life of between 2 and 4 years..

4. Financing the Budget

Equation (2) states that fluctuations in a ratio of government liabilities to a version of the primary deficit must be associated with fluctuations in future deficits, money creation, real interest rates, inflation and real GDP growth such that the governments intertemporal budget constraint holds. The way in which variations in these future values are linked to the current debt/deficit ratio is given very precisely in our model and can be exploited to reveal how fiscal adjustment has been achieved empiricaly. Following in the footsteps of Campbell and Shiller (1988) we write a VAR forecasting system for our equations of interest and show the restrictions that our model places on this forecasting model. This will enable us to decompose fluctuations in government debt/deficit ratio into its constituent components.

Let $l_t = \beta_1 b_{t-1} + \beta_2 h_{t-1} + \beta_3 d_t$ (where β_1, β_2 and β_3 are defined by the parameters in (2)) and

$\mathbf{Z}_t = (l_t, \Delta d_t, \Delta h_t, r_{t-1}, \pi_t, \gamma_t)$.Assume that $\boldsymbol{Z}_t$ follows a VAR(p) process, i.e.

$$\boldsymbol{Z}_t = A_1\boldsymbol{Z}_{t-1} + A_2\boldsymbol{Z}_{t-2} + ... + A_p\boldsymbol{Z} + \varepsilon_t$$

where $A_k,\ k = 1, ..., p$ are 6×6 matrices, whose $ij-$ th element is $a_{k,ij}$. By defining $\mathbf{z}_t = \left(z_t',\ z_{t-1}',\ ...,\ z_{t-p+1}'\right)$ and $\boldsymbol{\varepsilon}_t = (\varepsilon_t',\ 0, ...,\ 0)$ we can rewrite this VAR(p) as a VAR(1) so that

$$\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \boldsymbol{\varepsilon}_t$$

where

$$\mathbf{A} = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

so that conditional expectations satisfy

$$E_t\mathbf{z}_{t+j} = \mathbf{A}^j\mathbf{z}_t$$

Defining a series of indicator variables such that

$$\mathbf{e}_l'\mathbf{z}_t = l_t,\ \mathbf{e}_{\Delta d}'\mathbf{z}_t = \Delta d_t,\ \mathbf{e}_r'\mathbf{z}_t = r_{t-1},\ \mathbf{e}_{\Delta h}'\mathbf{z}_t = \Delta h_t,\quad \mathbf{e}_\pi'\mathbf{z}_t = \pi_t,\ \mathbf{e}_b'\mathbf{z}_t = b_{t-1}$$

we can then rewrite (2) as

$$\begin{aligned} e_l'\mathbf{z}_t &= -\frac{1}{\mu_b}[(1-\mu_b) + (1-\mu_h)\phi]e_{\Delta d}'\sum_1^\infty(\frac{\mathbf{A}}{\mu_b})^j\mathbf{z}_t \\ &+\frac{\phi}{\mu_b}(1-\frac{\mu_h}{\mu_b})e_{\Delta h}'\sum_0^\infty(\frac{\mathbf{A}}{\mu_b})^j\mathbf{z}_t \\ &-(1-\frac{1}{\mu_b})e_r'\sum_0^\infty(\frac{A}{\mu_b})^j\mathbf{z}_t + (1-\frac{1}{\mu_b})\phi e_\pi'\sum_0^\infty(\frac{A}{\mu_b})^j\mathbf{z}_t + (1-\frac{1}{\mu_b})(1+\phi\frac{\mu_h}{\mu_b})e_\gamma'\sum_0^\infty(\frac{A}{\mu_b})^j\mathbf{z}_t \end{aligned}$$

This is simply a restatement of our key equation (2) but where we have replaced the expectation terms with conditional forecasts obtained from our AR representation for $\mathbf{z}_t$. Using this approach we can also decompose l_t into its component parts e.g $l_t = F_{\Delta d} + F_{\Delta h} + F_r + F_\pi + F_\gamma$ where

$$\begin{aligned} F_{\Delta d} &= -\frac{1}{\mu_b}[(1-\mu_b) + (1-\mu_h)\phi]e_{\Delta d}'\frac{A}{\mu_b}(I - \frac{A}{\mu_b})^{-1}\mathbf{z}_t \\ F_{\Delta h} &= \frac{\phi}{\mu_b}(1-\frac{\mu_h}{\mu_b})e_{\Delta h}'(I-\frac{A}{\mu_b})^{-1}\mathbf{z}_t \\ F_r &= -(1-\frac{1}{\mu_b})e_r'(I-\frac{A}{\mu_b})^{-1}\mathbf{z}_t \\ F_\pi &= (1-\frac{1}{\mu_b})\phi e_\pi'(I-\frac{A}{\mu_b})^{-1}\mathbf{z}_t \\ F_\gamma &= (1-\frac{1}{\mu_b})(1+\phi\frac{\mu_h}{\mu_b})e_\gamma'(I-\frac{A}{\mu_b})^{-1}\mathbf{z}_t \end{aligned}$$

and F_j denotes the projected contribution of j towards maintaining the intertemporal budget constraint. Given our assumption on $\mathbf{z}_t$ we also have the following restrictions imposed by our system

$$\begin{aligned} \boldsymbol{e}_l^{'} &= -\frac{1}{\mu_b}[(1-\mu_b)+(1-\mu_h)\phi]\boldsymbol{e}_{\Delta d}^{'}\frac{A}{\mu_b}(I-\frac{A}{\mu_b})^{-1}+\frac{\phi}{\mu_b}(1-\frac{\mu_h}{\mu_b})\boldsymbol{e}_{\Delta h}^{'}(I-\frac{A}{\mu_b})^{-1} \qquad ((3)) \\ & -(1-\frac{1}{\mu_b})\boldsymbol{e}_r^{'}(I-\frac{A}{\mu_b})^{-1}+(1-\frac{1}{\mu_b})\phi\boldsymbol{e}_\pi^{'}(I-\frac{A}{\mu_b})^{-1}+(1-\frac{1}{\mu_b})(1+\phi\frac{\mu_h}{\mu_b})\boldsymbol{e}_\gamma^{'}(I-\frac{A}{\mu_b})^{-1} \end{aligned}$$

This expression shows the restriction that our present value formula imposes on innovations to l_t

relative to innovations in deficits, money and growth adjusted real interest rates. It can therefore be used as a test for the adequacy of our intertemporal bduget constraint model and the forecasting system used for our variables. Figure 4 shows the value of l_t and our estimated series for $F_{\Delta d} + F_{\Delta h} + F_r + F_\pi + F_\gamma$ in the case of the US and shows that our forecasting model does an excellent job capturing the intertemporal budget constraint's restrictions. This is shown more formally by using a χ^2 test to evaluate whether (3) holds. For each case we find the restriction comfortably accepted at p- values well below 5%.

As shown by Cochrane (1992) regressions of F_j on l_t can be used as a form of variance decomposition to ascertain which of these 5 components is most important in driving fiscal adjustments to l_t[5].The results for our sample are shown in Table 4. They show clearly that the majority of shocks to the debt/deficit ratio are corrected by adjustments to the primary deficit ($F_{\Delta d}$) across all countries. Interest rate variations tend to have a negative influence e.g when the fiscal position worsens interest rates shift in an adverse way, although this effect is nowhere substantial. This finding confirms the results of Marcet and Scott (2005) that debt management plays a limited role in helping stabilise government's fiscal position.The contribution of inflation to fiscal adjustment is of a similar order of magnitude across countries - around 2%. This finding clearly constrasts with Table 1 where we showed that the inflation component of the nominal growth dividend accounted for a substantial proportion of debt sustainability. However, the difference in our analysis is that here we are looking at the role of different factors in restoring fiscal sustainability in the face of shocks to the fiscal position. These results suggest that although rising government debt does tend to result in higher inflation the effect is not a substantial one, with fiscal sustainability normally achieved through adjustments to the primary fiscal position.

5. FORECASTING INFLATION

The previous section suggested that inflation played a minor role in accounting for shifts in the fiscal position of governments However, this does not mean that fiscal influences on inflation need to be small. In this section we consider another implication of equation (2) - that l_t should be useful in predicing future inflation. In particular we look at the ability of l_t to predict future inflation at horizons from 1 to 20 years. We do this by first specifying an optimal forecasting equation for inflation. We do this by using lag selection criteria in a model where inflation depends on lagged values of inflation,

[5] A complication is that the five expectation terms on the RHS of (2) are not independent and so the variance decomposition cannot be performed uniquely unless further assumptions are made. Table 4 essentially assumes that these covariance effects are spread equally across all 5 terms. As a result we sometimes find negative "variance decompositions" which clearly is inappropriate.

nominal interest rates and GDP growth where we consider lags of up to 8 periods for each variable. Having arrived at an optimal model we then add l_{t-j} to gauge the additional explanatory power from our measure of fiscal imbalance. The results are shown in Table 5 where we quote the p-value for $l_{t-j}, j = 1, .., 20$. The results are consistent with Table 4 - fiscal measures have a very marginal impact on predicting inflation. The vast majority of lags are insignificant although in a few cases there is evidence of predictive ability at horizons of around 3-4 years. However the marginal contribution of l_t is fairly small - fiscal contribution is not large. Inflation would seem to have mainly non-fiscal causes.

Figure 5 shows estimates of coherence plots between inflation and l_t. It shows little evidence of any correlation at higher frequencies although it does suggest stronger linkages at the longer term frequencies.

6. Demographic Implications

To be done

7. Conclusion

This paper sought to apply a log linearised version of the intertemporal budget constraint to consider government's fiscal positions. It tried to answer three key questions i) is current fiscal policy sustainable? ii) how have OECD governments financed their fiscal deficits in recent decades and iii) what are the implications for inflation of rising deficits?

In answer to the first question we estimated for each country a measure of current fiscal imbalance - a ratio between current liabilities and the primary deficit. For all countries the current measure fo this imbalance was within the historical range of variation suggesting that current policies are sustainable. Using our version of the intertemporal budget constraint we analysed how in previous years governments had achieved fiscal balance. We found an overwhelming role for changes in the primary surplus with only a minor role for inflation, growth and interest rate effects. Further we also found that fiscal imbalances had only a very weak forecasting role for future inflation at nearly all horizons, with some mild evidence that fiscal imbalances could help predict inflation 3-4 years ahead.

The obvious implication of the above is that looming fiscal deficits caused by demographic shifts should not exert a substantial impact on inflation. Naturally care must be taken with that conclusion. Our results are based on a certain historical period and relatively minor increases in debt - inevitably any attempt at an econometric approach to evaluating the intertemporal budget constraint is vulnerable to time dependence and non-stationarity. This caveat was emphasised in our final section where we used the intertemporal budget constraint and forecasts of future primary deficits to evaluate the impact on inflation were all fiscal adjustment to occur through this channel. In this case the impact on inflation was striking. In other words, if fiscal adjustment in the future is not achieved through similar methods as in the past then demographic change could have significant inflation implications.

8. Data Appendix

Notes on data sources for the UK and US can be found in Giannitsarou and Scott (2006). For the remaining countries details are as below.The following abbreviations are used:

GFD - Global Financial Data
IFS - International Financial Statistics (IMF)
OECD-EO - OECD Economic Outlook Database
OECD-CGD - OECD Central Government Debt Statistics
HSoC - Historical Statistics of Canada (Statistics Canada)
DI - DataInsight

GDP, Prices and Inflation

Country	variable	sample	source	ID/Specification
CAN	real GDP		GFD	GDPCCANM
	nom. GDP			GDPCANM
JAP	nom. GDP	1955-2005	IFS	15899B.CZF
	Deflator			15899BIRZF
JAP, ITA, GER	nom. GDP	1960-2005	OECD-EO	
	Deflator			

The implicit GDP deflator is used as the price index. (Gross) inflation is then obtained as the annual rate of change of the index.

Base Money

For Canada, Italy, and Japan, base money is used. For Germany, it is the national definition of M1 (currency in circulation plus overnight deposits).

Country	sample	source	ID/Specification
CAN	1926-1954	HSoC	J69+J71
	1955-2005	DI	MBASENS@CN
GER		DI	M1@EURNS@GY
ITA	1960-1990	Fratianni (2005), p49ff	col BP
	1991-2005	Banca d'Italia	
JAP		DI	MBASENS@JP

Government receipts and expenditure

All government expenditure data is net of interest service. Revenues are net of interest receipts for Germany, Italy, and Japan, but not for Canada. The primary deficit is expressed as net expenditure minus (net) receipts.

Country		sample	source	ID/Spec.
CAN	Receipts, expenditure	1926-1965	HSoC	F109, F116
	Receipts, expenditure	1966-2005	DI	REVG@CN, EXG@CN
	Interest			EXGCDINS@CN
GER, ITA, JAP	Receipts, expenditure		OECD-EO	
	Interest			

Government debt and market values

Market values are approximated for central government marketable debt, which is from OECD-CGD. For the periods before these data are available, the last available share of marketable in total debt was used to obtain marketable debt.

The price of government debt is approximated as

$$p_t = \frac{1 + NC}{1 + NI} \quad ,$$

where N is the average term to maturity of outstanding government securities, C is the average coupon rate, and I is the average market yield.

Data on average terms to maturity and average yields is from OECD-CGD. If no average term to maturity was available, average maturities were used. For earlier periods, the last average maturity available was taken. If average yields were unavailable, yields are constant maturity benchmark yields (from GFD). For a given year, the benchmark yield closest to the average term to maturity of that year was applied.

Average coupon data is approximated as the ratio of gross interest service to gross government debt, more precisely, $C_t = \frac{Interest_{t+1}}{DEBT_t}$.

9. Unit Root Tests

C means constant included. T means Trend included

9.1. Expenditure/GDP.

Country	sample	test	statistic	5% CV	verdict
CAN	1960-2005	ADF(1), C	-2.11	-2.86	UR
		KPSS(4)	0.748	0.463	UR
		KPSS(4) T	0.217	0.146	UR
GER	1960-2005	ADF(0)	-1.82	-1.94	UR
		KPSS(4)	0.60	0.46	UR
		KPSS(4) T	0.198	0.146	UR
ITA	1960-2005	ADF(2) C	-2.67	-2.86	UR
		ADF(2) C T	-1.36	-3.41	UR
		KPSS(4)	0.92	0.463	UR
		KPSS(4) T	0.236	0.146	UR
JAP	1960-2005	ADF(3)	-1.31	1.94	UR
		ADF(3) C T	-2.455	-3.41	UR
		KPSS(4)	0.941	0.463	UR
		KPSS(4)	0.137	0.146	T stationary
UK	1960-2005	ADF(9) C	-1.604	-2.86	UR
		ADF(10) C T	-3.745	-3.41	T stationary
		KPSS(4)	0.398	0.463	stationary
		KPSS(4) T	0.204	0.146	UR
US	1960-2005	ADF(7) C	-1.628	-2.86	UR
		ADF(6) C T	-1.993	-3.41	T stationary
		KPSS(4)	0.193	0.463	stationary
		KPSS(4) T	0.194	0.146	UR

10. Revenue/GDP

Country	**sample**	**test**	**statistic**	**5% CV**	**verdict**
CAN	1960-2005	ADF(0)	-1.359	-1.94	UR
		KPSS(4)	0.915	0.463	UR
		KPSS(4) T	0.155	0.146	UR
GER	1960-2005	ADF(1) C	-2.49	-2.86	UR
		KPSS(4)	0.713	0.46	UR
		KPSS(4) T	0.210	0.146	UR
ITA	1960-2005	ADF(0)	-2.43	-1.94	stationary
		KPSS(4)	0.969	0.463	UR
		KPSS(4) T	0.121	0.146	UR
JAP	1960-2005	ADF(0)	-2.44	1.94	stationary
		KPSS(4)	0.871	0.463	UR
		KPSS(4)	0.215	0.146	UR
UK	1960-2005	ADF(1) C	-2.95	-2.86	stationary
		KPSS(4)	0.186	0.463	stationary
		KPSS(4) T	0.184	0.146	UR
US	1960-2005	ADF(1) C	-3.73	-2.86	stationary
		KPSS(4)	0.141	0.463	stationary
		KPSS(4) T	0.045	0.146	T stationary

10.1. Debt/GDP.

Country	**sample**	**test**	**statistic**	**5% CV**	**verdict**
CAN	1960-2005	ADF(8) C	-3.45	-2.86	stationary
		ADF(8) C T	-3.91	-3.41	T stationary
		KPSS(4)	0.718	0.463	UR
		KPSS(4) T	0.159	0.146	UR
GER	1960-2005	ADF(1)	-2.28	-1.94	stationary
		ADF(1) C T	-2.44	-3.41	UR
		KPSS(4)	0.994	0.46	UR
		KPSS(4) T	0.150	0.146	UR
ITA	1960-2005	ADF(1)	-2.11	-1.94	stationary
		KPSS(4)	0.959	0.463	UR
		KPSS(4) T	0.151	0.146	UR
JAP	1960-2005	ADF(1)	-3.31	-1.94	stationary
		ADF(3) C T	-2.27	-3.41	UR
		KPSS(4)	0.969	0.463	UR
		KPSS(4) T	0.163	0.146	UR
UK	1960-2005	ADF(2) C	-2.51	-2.86	UR
		KPSS(4)	0.581	0.463	UR
		KPSS(4) T	0.204	0.146	UR
US	1960-2005	ADF(1) C	-1.79	-2.86	UR
		ADF(1) C T	-2.44	-3.41	UR
		KPSS(4)	0.381	0.463	stationary
		KPSS(4) T	0.146	0.146	UR

10.2. Money.

Country	sample	test	statistic	5% CV	verdict
CAN	1960-2005	ADF(1) C	-1.44	-2.86	UR
		KPSS(4)	0.971	0.463	UR
		KPSS(4) T	0.133	0.146	T stationary
GER	1960-2005	ADF(1) C	1.39	-2.86	UR
		ADF(1) C T	-1.11	-3.41	UR
		KPSS(4)	0.839	0.46	UR
		KPSS(4) T	0.253	0.146	UR
ITA	1960-2005	ADF(0)	1.41	-1.94	UR
		ADF(4) C T	-3.61	-3.41	T stationary
		KPSS(4)	0.880	0.463	UR
		KPSS(4) T	0.143	0.146	T stationary
JAP	1960-2005	ADF(1)	-1.73	-1.94	UR
		KPSS(4)	0.590	0.463	UR
		KPSS(4) T	0.182	0.146	UR
UK	1960-2005	ADF(5) C	-0.43	-1.94	UR
		KPSS(4)	0.940	0.463	UR
		KPSS(4) T	0.203	0.146	UR
US	1960-2005	ADF(1) C	-2.08	-2.86	UR
		ADF(0) C T	-0.45	-3.41	UR
		KPSS(4)	0.666	0.463	UR
		KPSS(4) T	0.252	0.146	UR

10.3. Real Return.

Country	sample	test	statistic	5% CV	verdict
CAN	1960-2005	ADF(0) C	-5.67	-2.86	stationary
		KPSS(4)	0.398	0.463	stationary
GER	1960-2005	ADF(0) C	-4.62	-2.86	stationary
		KPSS(4)	0.492	0.463	??
		KPSS(4) T	0.072	0.146	T stationary
ITA	1960-2005	ADF(2)	-1.934	-1.94	??
		ADF(4) C T	-4.44	-3.41	T stationary
		KPSS(4)	0.484	0.463	??
		KPSS(4) T	0.11	0.146	T stationary
JAP	1960-2005	ADF(0) c	-6.06	–2.86	stationary
		KPSS(4)	0.420	0.463	stationary
UK	1960-2005	ADF(0) C	-7.53	-2.86	stationary
		KPSS(4)	0.309	0.463	stationary
US	1960-2005	ADF(2)	-2.39	-1.94	stationary
		KPSS(4)	0.301	0.463	stationary

10.4. inflation.

Country	**sample**	**test**	**statistic**	**5% CV**	**verdict**
CAN	1960-2005	ADF(1) C	-2.34	-2.86	UR
		KPSS(4)	0.360	0.463	stationary
GER	1960-2005	ADF(7)	-0.75	-1.94	UR
		ADF(5) C T	-3.27	-3.41	UR
		KPSS(4)	0.654	0.463	UR
		KPSS(4) T	0.118	0.146	T stationary
ITA	1960-2005	ADF(3)	-1.04	-1.94	UR
		KPSS(4)	0.280	0.463	stationary
JAP	1960-2005	ADF(0)	-2.05	-1.94	stationary
		KPSS(4)	0.764	0.463	UR
		KPSS(4) T	0.098	0.146	T stationary
UK	1960-2005	ADF(0) C	-2.27	-2.86	UR
		KPSS(4)	0.262	0.463	stationary
US	1960-2005	ADF(0)	-0.70	-1.94	UR
		KPSS(4)	0.24	0.463	stationary

10.5. Growth.

Country	**sample**	**test**	**statistic**	**5% CV**	**verdict**
CAN	1960-2005	ADF(0) C	-4.62	-2.86	stationary
		KPSS(4)	0.318	0.463	stationary
GER	1960-2005	ADF(0)	-4.48	-2.86	stationary
		KPSS(4)	0.240	0.463	stationary
ITA	1960-2005	ADF(0) C	-4.31	-2.86	stationary
		ADF(0) C T	-6.30	-3.41	T stationary
		KPSS(4)	0.887	0.463	UR
		KPSS(4) T	0.095	0.146	T stationary
JAP	1960-2005	ADF(2)	-1.82	-1.94	UR
		KPSS(4)	0.740	0.463	UR
		KPSS(4) T	0.098	0.146	T stationary
UK	1960-2005	ADF(1) C	-5.26	-2.86	stationary
		KPSS(4)	0.117	0.463	stationary
US	1960-2005	ADF(8) C	-3.64	-2.86	stationary
		KPSS(4)	0.205	0.463	stationary

Table 1 : Historical Sources of Debt Variation

Country	sample	tdef	pdef	interest	nom. gr	real gr	inf	$\Delta\frac{Debt}{GDP}$
US	1960-2005	0.021	0.000	0.021	0.019	0.010	0.009	0.003
UK	1960-2005	0.017	-0.021	0.039	0.041	0.014	0.027	-0.014
GER	1960-2005	0.016	-0.001	0.021	0.009	0.005	0.004	0.064
JAP	1960-2005	0.018	-0.008	0.024	0.011	0.008	0.003	0.080
ITA	1960-2005	0.064	0.021	0.043	0.059	0.015	0.044	0.025
CAN	1960-2005	0.029	-0.005	0.035	0.023	0.011	0.012	0.013

Table shows sample averages for each country for total deficit/GDP, primary deficit/GDP, interest payments/GDP, nominal GDP growth, real GDP growth and inflation.The final column shows the average per period change in the debt/GDP ratio

Table 2 : Estimates of Equilibrium Relationship Debt and Deficits

	US	Canada	Germany	Japan	UK	Italy
ϕ	0.302	0.174	1.060	0.612	0.103	0.267
γ	1.056	1.064	1.065	1.055	1.025	1.081
μ_b	1.020	1.025	1.011	1.020	1.002	1.044
μ_h	0.966	0.964	0.949	0.966	0.978	0.966
λ_g	-6.968	-15.650	2.362	-2564	-56.529	-3.073
λ_t	7.982	16.650	-1.369	2565	57.523	4.073
β_b	0.019	0.025	0.011	0.019	0.002	0.043
β_h	-0.011	-0.006	-0.053	-0.022	-0.002	-0.009
β_g	0.065	0.293	0.101	-1.412	0.014	0.105
β_t	-0.074	-0.311	-0.058	1.418	-0.014	-0.139
$\beta_b + \beta_h = 1$	0.031	0.047	0.014	0.033	0.061	0.084

The first row reports the sample average of H/B and the second row reports the sample average of nominal GDP growth. μ_b and μ_t are estimated as are $\lambda_g$and λ_t(subject to the restriction that $\lambda_g + \lambda_t = 1$).The final rows show estimates of coefficients for defining $l_t = \beta_b b_t + \beta_h h_t + \beta_g g_t + \beta_t t_t$

Table 3 - Dynamics of Fiscal Adjustment

	US	Canada	Germany	Japan	UK	Italy
Min	-0.0314	-0.065	-0.050	-0.045	-0.055	-0.049
Max	0.025	0.053	0.053	0.060	0.049	0.062
Std Dev	0.015	0.027	0.022	0.027	0.024	0.026
Sum AR	0.758	0.860	0.718	0.608	0.756	0.590
Unit Root	0.042	0.031	0.10	0.046	0.054	0.027
25%	0.96	3.3	0.93	1.3	0.21	3
50%	2.5	4.1	2.1	2.1	2.5	4.1
75%	5.0	5.2	4.1	2.9	4.96	6.9

First row shows minimum value of l_t over sample period, second row the maximum value. Third row is the standard deviation of l_t while the fourth row shows the sum of the AR coefficients when l_t is modelled as an AR(P) process where P is chosen optimally using AIC criteria. The next row is the p-value from an ADF test that l_t is a unit root process. The last three rows show the number of periods it takes l_t to adjust by 25%, 50% and 75% respectively to a shock to its value.

Table 4 : Variance Decomposition of Fiscal Adjustments

	$F_{\Delta d}$	$F_{\Delta h}$	F_r	F_π	F_γ	Test
US	0.916	-0.018	0.028	0.020	0.099	0.031
Canada	0.985	0.010	-0.013	0.002	-0.018	0.022
Germany	0.930	0.053	-0.021	0.042	0.008	0.045
Japan	1.071	-0.066	0.042	0.066	0.059	0.017
UK	0.912	0.113	-0.141	0.025	-0.003	0.038
Italy	0.881	0.070	-0.054	0.068	0.037	0.062

Table 5 Predicting Inflation

	US	Canada	Germany	Japan	UK	Italy
1	0.04	0.08	0.21	0.13	0.23	0.16
2	0.07	0.05	0.26	0.11	0.31	0.13
3	0.08	0.07	0.32	0.07	0.17	0.12
4	0.13	0.11	0.21	0.06	0.13	0.08
5	0.09	0.17	0.17	0.08	0.04	0.04
6	0.07	0.32	0.16	0.12	0.03	0.15
7	0.21	0.25	0.35	0.15	0.05	0.31
8	0.32	0.21	0.49	0.16	0.11	0.24
9	0.36	0.42	0.45	0.21	0.19	0.42
10	0.48	0.56	0.26	0.21	0.23	0.46
11	0.63	0.54	0.32	0.19	0.35	0.53
12	0.54	0.59	0.54	0.26	0.46	0.41
13	0.52	0.58	0.56	0.29	0.32	0.36
14	0.59	0.65	0.63	0.32	0.41	0.39
15	0.73	0.78	0.78	0.33	0.53	0.55
16	0.86	0.79	0.74	0.34	0.52	0.45
17	0.88	0.82	0.72	0.30	0.46	0.34
18	0.75	0.91	0.76	0.26	0.64	0.46
19	0.82	0.96	0.73	0.45	0.71	0.74
20	0.95	0.99	0.74	0.54	0.55	0.65

Table shows p-values of significance of l_{t-j} (where j is listed in the first column) in a forecasting equation for inflation containing lagged values of inflation, interest rates and GDP growth.

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Appendix - Log Linearising Intertemporal Budget Constraint

For easy reference, the following tables give the notation and definitions we use throughout the paper. Upper case variables denote macro variables over GDP, or gross rates of change. Upper case letters with bars denote these variables deflated by "aggregate wealth". Lower case letters denote logs of the upper case letter variables. Lower case letters with a bar denote deviations of the upper case barred variables from their steady states. The last table gives the definitions of several auxiliary parameters.

variable	definition	st. state
G_t	government spending over GDP	
T_t	tax revenues over GDP	
B_t	debt over GDP	
H_t	seignorage over GDP	
W_t	aggregate wealth	
P_t	price index	
Y_t	real GDP	
$R_t = 1 + r_t$	gross real interest rate	$R = 1 + r$
$\Upsilon_t = 1 + i_t$	gross nominal interest rate	$\Upsilon = 1 + i$
Ω_t	$= \frac{W_t}{W_{t-1}}$	$\Omega = 1 + \omega$
$\Pi_t = 1 + \pi_t$	$= \frac{P_t}{P_{t-1}}$	$\Pi = 1 + \pi$
$Q_t = 1 + \gamma_t$	$= \frac{Y_t}{Y_{t-1}}$	$Q = 1 + \gamma$

variable	definition	st. state
$\bar{G}_t$	$= G_t/W_t$	$\bar{G}$
$\bar{T}_t$	$= T_t/W_t$	$\bar{T}$
$\bar{B}_t$	$= B_t/W_t$	$\bar{B}$
$\bar{H}_t$	$= H_t/W_y$	$\bar{H}$
w_t	$= \ln W_t$	
g_t	$= \ln G_t$	
τ_t	$= \ln T_t$	
b_t	$= \ln B_t$	
h_t	$= \ln H_t$	
d_t	$= \lambda_g g_t + \lambda_\tau \tau_t$ (weighted primary deficit)	

variable	definition
$\bar{w}_t$	$= \ln(\Omega_t/\Omega_{t-1}) = w_t - w_{t-1} - \ln\Omega$
$\bar{g}_t$	$= \ln\left(\bar{G}_t/\bar{G}\right) = g_t - w_t - \ln\bar{G}$
$\bar{\tau}_t$	$= \ln\left(\bar{T}_t/\bar{T}\right) = \tau_t - w_t - \ln\bar{T}$
$\bar{b}_t$	$= \ln\left(\bar{B}_t/\bar{B}\right) = b_t - w_t - \ln\bar{B}$
$\bar{h}_t$	$= \ln\left(\bar{H}_t/\bar{H}\right) = h_t - w_t - \ln\bar{H}$
υ_t	$= \ln(\Upsilon_t/\Upsilon) = \ln(1+i_t) - \ln\Upsilon \approx i_t - i$
x_t	$= \ln(Q_t/Q) = \ln(1+\gamma_t) - \ln Q \approx \gamma_t - \gamma$
φ_t	$= \ln(\Pi_t/\Pi) = \ln(1+\pi_t) - \ln\Pi \approx \pi_t - \pi$

variable	definition	data
μ_b	$= \frac{\Upsilon}{\Pi Q\Omega}$	well-defined
μ_h	$= \frac{1}{\Pi Q\Omega}$	well-defined
λ_g	$= \frac{\bar{G}}{\bar{G}-\bar{T}}$	well-defined
λ_τ	$= -\frac{\bar{T}}{\bar{G}-\bar{T}}$	well-defined
ϕ	$= \frac{\bar{H}}{\bar{B}}$	well-defined
m	$= (1-\mu_b)\,\bar{B} + (1-\mu_h)\,\bar{H}$	well-defined
κ	summary of constants that we can ignore	

The budget constraint for the government after having adjusted with GDP and prices can be written as

$$G_t - T_t = B_t - \frac{\Upsilon_{t-1}}{\Pi_t Q_t} B_{t-1} + H_t - \frac{1}{\Pi_t Q_t} H_{t-1}.$$

Dividing through with aggregate wealth, W_t, we get

$$\frac{G_t}{W_t} - \frac{T_t}{W_t} = \frac{B_t}{W_t} - \frac{\Upsilon_{t-1}}{\Pi_t Q_t \Omega_t}\frac{B_{t-1}}{W_{t-1}} + \frac{H_t}{W_t} - \frac{1}{\Pi_t Q_t \Omega_t}\frac{H_{t-1}}{\Omega_{t-1}},$$

i.e.

$$\bar{G}_t - \bar{T}_t = \bar{B}_t - \frac{\Upsilon_{t-1}}{\Pi_t Q_t \Omega_t}\bar{B}_{t-1} + \bar{H}_t - \frac{1}{\Pi_t Q_t \Omega_t}\bar{H}_{t-1}.$$

In this last expression, all variables are by assumption stationary. Thus we can log-linearise the expression. To do this, we rewrite it as

$$\Omega_t \bar{G}_t - \Omega_t \bar{T}_t = \Omega_t \bar{B}_t - \frac{\Upsilon_{t-1}}{\Pi_t Q_t}\bar{B}_{t-1} + \Omega_t \bar{H}_t - \frac{1}{\Pi_t Q_t}\bar{H}_{t-1},$$

We use the approximation

$$\exp(z) \approx z + 1$$

and the steady state relationship

$$\begin{aligned} \bar{G} - \bar{T} &= (1-\mu_b)\,\bar{B} + (1-\mu_h)\,\bar{H} \equiv m \quad \text{or} \\ \left(\bar{G} - \bar{T}\right) - \left(\bar{B} + \bar{H}\right) &= -\left(\mu_b \bar{B} + \mu_h \bar{H}\right) \end{aligned}$$

to approximate (XX) by

$$\begin{aligned}\Omega\bar{G}\left(\bar{w}_t+\bar{g}_t\right)-\Omega\bar{G}\left(\bar{w}_t+\bar{\tau}_t\right) &= \Omega\bar{B}\left(w_t+\bar{b}_t\right)-\frac{\Upsilon\bar{B}}{\Pi Q}\left(\upsilon_{t-1}-\varphi_t-x_t+\bar{b}_{t-1}\right)\\ &\quad+\Omega\bar{H}\left(w_t+\bar{h}_t\right)-\frac{\bar{H}}{\Pi Q}\left(-\varphi_t-x_t+\bar{h}_{t-1}\right).\end{aligned}$$

Dividing through with Ω and collecting we get

$$\left[\left(\bar{G}-\bar{T}\right)-\left(\bar{B}+\bar{H}\right)\right]\bar{w}_t+\bar{G}\bar{g}_t-\bar{T}\bar{\tau}_t=\bar{B}\bar{b}_t+\bar{H}\bar{h}_t-\left[\mu_b\bar{B}\left(\upsilon_{t-1}-\varphi_t-x_t+\bar{b}_{t-1}\right)+\mu_h\bar{H}\left(-\varphi_t-x_t+\bar{h}_{t-1}\right)\right].$$

By using the steady state relationship this becomes

$$-\left(\mu_b\bar{B}+\mu_h\bar{H}\right)\bar{w}_t+\bar{G}\bar{g}_t-\bar{T}\bar{\tau}_t=\bar{B}\bar{b}_t+\bar{H}\bar{h}_t-\left[\mu_b\bar{B}\left(\upsilon_{t-1}-\varphi_t-x_t+\bar{b}_{t-1}\right)+\mu_h\bar{H}\left(-\varphi_t-x_t+\bar{h}_{t-1}\right)\right].$$

Next, we susbtitute the barred variables and ignore any (steady state) constants. We also replace υ_t, φ_t and x_t with their approximations, and we ignore all constants.

$$\begin{aligned}&-\left(\mu_b\bar{B}+\mu_h\bar{H}\right)\left(w_t-w_{t-1}\right)+\bar{G}\left(g_t-w_t\right)-\bar{T}\left(\tau_t-w_t\right)\\ =\; &\bar{B}\left(b_t-w_t\right)+\bar{H}\left(h_t-w_t\right)-\left[\mu_b\bar{B}\left(i_{t-1}-\pi_t-\gamma_t+b_{t-1}-w_{t-1}\right)+\mu_h\bar{H}\left(-\pi_t-\gamma_t+h_{t-1}-w_{t-1}\right)\right].\end{aligned}$$

Now all the terms containing w_t or w_{t-1} cancel out (by using the steady state relation), so what we have left is

$$\bar{G}g_t-\bar{T}\tau_t=\bar{B}b_t+\bar{H}h_t-\left[\mu_b\bar{B}\left(i_{t-1}-\pi_t-\gamma_t+b_{t-1}\right)+\mu_h\bar{H}\left(-\pi_t-\gamma_t+h_{t-1}\right)\right].$$

Dividing through with $\bar{G}-\bar{T}$ and using the approximation

$$r_{t-1}\approx i_{t-1}+\pi_t$$

we get

$$d_t=\frac{\bar{B}}{m}b_t-\frac{\mu_b\bar{B}}{m}b_{t-1}+\frac{\bar{H}}{m}h_t-\frac{\mu_h\bar{H}}{m}h_{t-1}-\frac{\mu_b\bar{B}}{m}r_{t-1}+\frac{\mu_h\bar{H}}{m}\pi_t+\frac{\mu_b\bar{B}+\mu_h\bar{H}}{m}\gamma_t$$

which we can rewrite as

$$d_t=\frac{\bar{B}}{m}\left(b_t-\mu_b b_{t-1}\right)+\frac{\bar{H}}{m}\left(h_t-\mu_h h_{t-1}\right)-\frac{\bar{B}}{m}\mu_b r_{t-1}+\frac{\bar{H}}{m}\mu_h\pi_t+\frac{\mu_b\bar{B}+\mu_h\bar{H}}{m}\gamma_t$$

- Isolate b_{t-1}

$$\begin{aligned}\frac{m}{\bar{B}}d_t &= \left(b_t-\mu_b b_{t-1}\right)+\frac{\bar{H}}{\bar{B}}\left(h_t-\mu_h h_{t-1}\right)-\mu_b r_{t-1}+\frac{\bar{H}}{\bar{B}}\mu_h\pi_t+\left(\frac{\mu_b\bar{B}+\mu_h\bar{H}}{\bar{B}}\right)\gamma_t\\ \frac{m}{\bar{B}}d_t &= \left(b_t-\mu_b b_{t-1}\right)+\phi\left(h_t-\mu_h h_{t-1}\right)-\mu_b r_{t-1}+\phi\mu_h\pi_t+\left(\mu_b+\mu_h\phi\right)\gamma_t\\ \frac{m}{\bar{B}}\frac{1}{\mu_b}d_t &= \frac{1}{\mu_b}b_t-b_{t-1}+\frac{\phi}{\mu_b}\left(h_t-\mu_h h_{t-1}\right)-r_{t-1}+\phi\frac{\mu_h}{\mu_b}\pi_t+\left(1+\phi\frac{\mu_h}{\mu_b}\right)\gamma_t\\ b_{t-1} &= \frac{1}{\mu_b}b_t-\frac{1}{\mu_b}\frac{m}{\bar{B}}d_t+\frac{\phi}{\mu_b}\left(h_t-\mu_h h_{t-1}\right)-r_{t-1}+\phi\frac{\mu_h}{\mu_b}\pi_t+\left(1+\phi\frac{\mu_h}{\mu_b}\right)\gamma_t\\ b_{t-1} &= \frac{1}{\mu_b}b_t-\frac{1}{\mu_b}\frac{m}{\bar{B}}d_t+\frac{\phi}{\mu_b}s_t-r_{t-1}+\phi\frac{\mu_h}{\mu_b}\pi_t+\left(1+\phi\frac{\mu_h}{\mu_b}\right)\gamma_t\end{aligned}$$

- Substitute forward

$$\begin{aligned} b_{t-1} &= \frac{1}{\mu_b} b_t - \frac{1}{\mu_b}\frac{m}{\bar{B}} d_t + \frac{\phi}{\mu_b} s_t - r_{t-1} + \phi\frac{\mu_h}{\mu_b}\pi_t + \left(1+\phi\frac{\mu_h}{\mu_b}\right)\gamma_t \\ &= \left(\frac{1}{\mu_b}\right)^T b_{t+T-1} + \sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j \left\{-\frac{1}{\mu_b}\frac{m}{\bar{B}} d_{t+j} + \frac{\phi}{\mu_b} s_{t+j} - r_{t+j-1} + \phi\frac{\mu_h}{\mu_b}\pi_{t+j} + \left(1+\phi\frac{\mu_h}{\mu_b}\right)\gamma_{t+j}\right\} \end{aligned}$$

- Let

$$X = \sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j \left\{\frac{\phi}{\mu_b} s_{t+j} - r_{t+j-1} + \phi\frac{\mu_h}{\mu_b}\pi_{t+j} + \left(1+\phi\frac{\mu_h}{\mu_b}\right)\gamma_{t+j}\right\}$$

- Then

$$\begin{aligned} b_{t-1} &= \left(\frac{1}{\mu_b}\right)^T b_{t+T-1} - \sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j \frac{1}{\mu_b}\frac{m}{\bar{B}} d_{t+j} + X \\ b_{t-1} - X &= \left(\frac{1}{\mu_b}\right)^T b_{t+T-1} - \sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j \frac{1}{\mu_b}\frac{m}{\bar{B}} d_{t+j} \\ b_{t-1} - X &= \left(\frac{1}{\mu_b}\right)^T b_{t+T-1} - \frac{1}{\mu_b}\frac{m}{\bar{B}} d_t - \frac{1}{\mu_b^2}\frac{m}{\bar{B}} d_{t+1} - ... - \left(\frac{1}{\mu_b}\right)^T \frac{m}{\bar{B}} d_{t+T-1} \quad \text{(A)} \\ \frac{1}{\mu_b}(b_{t-1} - X) &= \left(\frac{1}{\mu_b}\right)^{T+1} b_{t+T-1} - \frac{1}{\mu_b^2}\frac{m}{\bar{B}} d_t - \frac{1}{\mu_b^3}\frac{m}{\bar{B}} d_{t+1} - ... - \left(\frac{1}{\mu_b}\right)^{T+1} \frac{m}{\bar{B}} d_{t+T-1} \quad \text{(B)} \end{aligned}$$

- Subtract [B] from [A]

$$\begin{aligned} (b_{t-1} - X)\left(1 - \frac{1}{\mu_b}\right) &= \left[\left(\frac{1}{\mu_b}\right)^T b_{t+T-1} - \left(\frac{1}{\mu_b}\right)^{T+1} b_{t+T-1}\right] \\ &\quad - \frac{1}{\mu_b}\frac{m}{\bar{B}} d_t - \frac{1}{\mu_b^2}\frac{m}{\bar{B}} d_{t+1} + \frac{1}{\mu_b^2}\frac{m}{\bar{B}} d_t \\ &\quad - \frac{1}{\mu_b^3}\frac{m}{\bar{B}} d_{t+2} + \frac{1}{\mu_b^3}\frac{m}{\bar{B}} d_{t+1} - ... \\ &\quad - \left(\frac{1}{\mu_b}\right)^T \frac{m}{\bar{B}} d_{t+T-1} + \left(\frac{1}{\mu_b}\right)^T \frac{m}{\bar{B}} d_{t+T-2} \\ &\quad + \left(\frac{1}{\mu_b}\right)^{T+1} \frac{m}{\bar{B}} d_{t+T-1} \end{aligned}$$

- Which becomes

$$\begin{aligned}(b_{t-1}-X)\left(1-\frac{1}{\mu_b}\right) &= \left[\left(\frac{1}{\mu_b}\right)^T b_{t+T-1}-\left(\frac{1}{\mu_b}\right)^{T+1} b_{t+T-1}\right] \\ &\quad -\frac{1}{\mu_b}\frac{m}{\bar{B}}d_t+\left(\frac{1}{\mu_b}\right)^{T+1}\frac{m}{\bar{B}}d_{t+T-1} \\ &\quad -\frac{1}{\mu_b^2}\frac{m}{\bar{B}}\Delta d_{t+1}-\frac{1}{\mu_b^3}\frac{m}{\bar{B}}\Delta d_{t+2}-...-\left(\frac{1}{\mu_b}\right)^T\frac{m}{\bar{B}}\Delta d_{t+T-1}\end{aligned}$$

- Or

$$\begin{aligned}(b_{t-1}-X)\left(1-\frac{1}{\mu_b}\right) &= \left(1-\frac{1}{\mu_b}\right)\left(\frac{1}{\mu_b}\right)^T b_{t+T-1} \\ &\quad -\frac{1}{\mu_b}\frac{m}{\bar{B}}d_t+\left(\frac{1}{\mu_b}\right)^{T+1}\frac{m}{\bar{B}}d_{t+T-1} \\ &\quad -\frac{1}{\mu_b^2}\frac{m}{\bar{B}}\Delta d_{t+1}-\frac{1}{\mu_b^3}\frac{m}{\bar{B}}\Delta d_{t+2}-...-\left(\frac{1}{\mu_b}\right)^T\frac{m}{\bar{B}}\Delta d_{t+T-1}\end{aligned}$$

$$(b_{t-1}-X)\left(1-\frac{1}{\mu_b}\right)+\frac{1}{\mu_b}\frac{m}{\bar{B}}d_t=\left(\frac{1}{\mu_b}\right)^T\left[\left(1-\frac{1}{\mu_b}\right)b_{t+T-1}+\frac{1}{\mu_b}\frac{m}{\bar{B}}d_{t+T-1}\right]-\frac{1}{\mu_b}\frac{m}{\bar{B}}\sum_{j=1}^{T-1}\left(\frac{1}{\mu_b}\right)^j\Delta d_{t+j}$$

$$(b_{t-1}-X)\left(1-\frac{1}{\mu_b}\right)=\underbrace{-\frac{1}{\mu_b}\frac{m}{\bar{B}}d_t+\left(\frac{1}{\mu_b}\right)^T\left[\left(1-\frac{1}{\mu_b}\right)b_{t+T-1}+\frac{1}{\mu_b}\frac{m}{\bar{B}}d_{t+T-1}\right]-\frac{1}{\mu_b}\frac{m}{\bar{B}}\sum_{j=1}^{T-1}\left(\frac{1}{\mu_b}\right)^j\Delta d_{t+j}}_{=Z}$$

$$\begin{aligned}b_{t-1}-X &= \left(1-\frac{1}{\mu_b}\right)^{-1}Z \\ b_{t-1}-\underbrace{X_1-\sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j\frac{\phi}{\mu_b}s_{t+j}}_{=X} &= \left(1-\frac{1}{\mu_b}\right)^{-1}Z \\ \underbrace{b_{t-1}-X_1-\left(1-\frac{1}{\mu_b}\right)^{-1}Z}_{=M} &= \sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j\frac{\phi}{\mu_b}s_{t+j} \\ M &= \sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j\frac{\phi}{\mu_b}s_{t+j}\end{aligned}$$

- We have

$$\begin{aligned}
M &= \sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j \frac{\phi}{\mu_b}\left(h_{t+j}-\mu_h h_{t+j-1}\right) \\
&= \frac{\phi}{\mu_b}\left(h_t-\mu_h h_{t-1}\right)+\frac{1}{\mu_b}\frac{\phi}{\mu_b}\left(h_{t+1}-\mu_h h_t\right) \\
&\quad +\frac{1}{\mu_b^2}\frac{\phi}{\mu_b}\left(h_{t+2}-\mu_h h_{t+1}\right)+...+\left(\frac{1}{\mu_b}\right)^{T-1}\frac{\phi}{\mu_b}\left(h_{t+T-1}-\mu_h h_{t+T-2}\right) \\
&= -\frac{\phi}{\mu_b}\mu_h h_{t-1}+\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right)h_t+\frac{1}{\mu_b}\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right)h_{t+1} \\
&\quad +...+\left(\frac{1}{\mu_b}\right)^{T-2}\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right)h_{t+T-2}+\left(\frac{1}{\mu_b}\right)^{T-1}\frac{\phi}{\mu_b}h_{t+T-1} \\
&= -\frac{\phi}{\mu_b}\mu_h h_{t-1}+\left(\frac{1}{\mu_b}\right)^{T-1}\frac{\phi}{\mu_b}h_{t+T-1}+\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right)\sum_{j=0}^{T-2}\left(\frac{1}{\mu_b}\right)^j h_{t+j}
\end{aligned}$$

- Divide both sides with μ_b

$$\frac{M}{\mu_b}=-\frac{\phi}{\mu_b}\frac{\mu_h}{\mu_b}h_{t-1}+\left(\frac{1}{\mu_b}\right)^{T}\frac{\phi}{\mu_b}h_{t+T-1}+\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right)\sum_{j=0}^{T-2}\left(\frac{1}{\mu_b}\right)^{j+1}h_{t+j}$$

- Subtract

$$\begin{aligned}
\left(1-\frac{1}{\mu_b}\right)M &= -\frac{\phi}{\mu_b}\mu_h h_{t-1}+\left(\frac{1}{\mu_b}\right)^{T-1}\frac{\phi}{\mu_b}h_{t+T-1}+\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right)\sum_{j=0}^{T-2}\left(\frac{1}{\mu_b}\right)^j h_{t+j} \\
&\quad +\frac{\phi}{\mu_b}\frac{\mu_h}{\mu_b}h_{t-1}-\left(\frac{1}{\mu_b}\right)^{T}\frac{\phi}{\mu_b}h_{t+T-1}-\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right)\sum_{j=0}^{T-2}\left(\frac{1}{\mu_b}\right)^{j+1}h_{t+j}
\end{aligned}$$

- Tidy up

$$\begin{aligned}
\left(1-\frac{1}{\mu_b}\right)M &= -\frac{\phi}{\mu_b}\mu_h\left(1-\frac{1}{\mu_b}\right)h_{t-1}+\left(\frac{1}{\mu_b}\right)^{T-1}\frac{\phi}{\mu_b}\left(1-\frac{1}{\mu_b}\right)h_{t+T-1} \\
&\quad +\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right)\sum_{j=0}^{T-2}\left(\frac{1}{\mu_b}\right)^j h_{t+j}-\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right)\sum_{j=0}^{T-2}\left(\frac{1}{\mu_b}\right)^{j+1}h_{t+j}
\end{aligned}$$

- Next

$$\begin{aligned}
\left(1-\frac{1}{\mu_b}\right)M &= -\frac{\phi}{\mu_b}\mu_h\left(1-\frac{1}{\mu_b}\right)h_{t-1}+\left(\frac{1}{\mu_b}\right)^{T-1}\frac{\phi}{\mu_b}\left(1-\frac{1}{\mu_b}\right)h_{t+T-1} \\
&\quad +\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right)\left[h_t+\frac{1}{\mu_b}\Delta h_{t+1}+\frac{1}{\mu_b^2}\Delta h_{t+2}+...+\frac{1}{\mu_b^{T-2}}\Delta h_{t+T-2}-\frac{1}{\mu_b^{T-1}}h_{t+T-2}\right]
\end{aligned}$$

- Then add and take away terms to get Δh_t and Δh_{t+T-1}

$$\begin{aligned} & \left(1-\frac{1}{\mu_b}\right) M+\frac{\phi}{\mu_b}\left(\mu_h-1\right) h_{t-1} \\ = & \left(\frac{1}{\mu_b}\right)^T \frac{\phi}{\mu_b}\left(\mu_h-1\right) h_{t+T-1}+\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right) \sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j \Delta h_{t+j} \end{aligned}$$

- Recall

$$\begin{aligned} M\left(1-\frac{1}{\mu_b}\right) &= b_{t-1}\left(1-\frac{1}{\mu_b}\right)-X_1\left(1-\frac{1}{\mu_b}\right)-Z \\ X_1 &= \sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j\left\{-r_{t+j-1}+\phi \frac{\mu_h}{\mu_b} \pi_{t+j}+\left(1+\phi \frac{\mu_h}{\mu_b}\right) \gamma_{t+j}\right\} \\ Z &= -\frac{1}{\mu_b} \frac{m}{\bar{B}} d_t+\left(\frac{1}{\mu_b}\right)^T\left[\left(1-\frac{1}{\mu_b}\right) b_{t+T-1}+\frac{1}{\mu_b} \frac{m}{\bar{B}} d_{t+T-1}\right]-\frac{1}{\mu_b} \frac{m}{\bar{B}} \sum_{j=1}^{T-1}\left(\frac{1}{\mu_b}\right)^j \Delta d_{t+j} \end{aligned}$$

- So that

$$\begin{aligned} & \left(1-\frac{1}{\mu_b}\right) b_{t-1}+\phi \frac{\mu_h}{\mu_b}\left(1-\frac{1}{\mu_h}\right) h_{t-1}+\frac{1}{\mu_b} \frac{m}{\bar{B}} d_t \\ = & \left(\frac{1}{\mu_b}\right)^T\left[\left(1-\frac{1}{\mu_b}\right) b_{t+T-1}+\phi \frac{\mu_h}{\mu_b}\left(1-\frac{1}{\mu_h}\right) h_{t+T-1}+\frac{1}{\mu_b} \frac{m}{\bar{B}} d_{t+T-1}\right] \\ & -\frac{1}{\mu_b} \frac{m}{\bar{B}} \sum_{j=1}^{T-1}\left(\frac{1}{\mu_b}\right)^j \Delta d_{t+j}+\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right) \sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j \Delta h_{t+j} \\ & +\left(1-\frac{1}{\mu_b}\right) \sum_{j=0}^{T-1}\left(\frac{1}{\mu_b}\right)^j\left\{-r_{t+j-1}+\phi \frac{\mu_h}{\mu_b} \pi_{t+j}+\left(1+\phi \frac{\mu_h}{\mu_b}\right) \gamma_{t+j}\right\} \end{aligned}$$

- We need to assume the transversality condition

$$\lim_{T \rightarrow \infty}\left(\frac{1}{\mu_b}\right)^T\left[\left(1-\frac{1}{\mu_b}\right) b_{t+T-1}+\phi \frac{\mu_h}{\mu_b}\left(1-\frac{1}{\mu_h}\right) h_{t+T-1}+\frac{1}{\mu_b} \frac{m}{\bar{B}} d_{t+T-1}\right]=0$$

  However from the budget constraint we know that this transversality condition will hold if we assume

- $\lim_{N \rightarrow \infty}\left(\frac{1}{\mu_b}\right)^N B_{t+N-1}=0$ and $\lim_{N \rightarrow \infty}\left(\frac{1}{\mu_b}\right)^N H_{t+N-1}=0$

- Therefore in the limit we have

$$\underbrace{\left(1-\frac{1}{\mu_b}\right)b_{t-1}+\phi\frac{\mu_h}{\mu_b}\left(1-\frac{1}{\mu_h}\right)h_{t-1}}_{=\,l_{t-1}}+\frac{1}{\mu_b}\left[(1-\mu_b)+(1-\mu_h)\,\phi\right]d_t=$$

$$=-\frac{1}{\mu_b}\left[(1-\mu_b)+(1-\mu_h)\,\phi\right]\sum_{j=1}^{\infty}\left(\frac{1}{\mu_b}\right)^j\Delta d_{t+j}$$

$$+\frac{\phi}{\mu_b}\left(1-\frac{\mu_h}{\mu_b}\right)\sum_{j=0}^{\infty}\left(\frac{1}{\mu_b}\right)^j\Delta h_{t+j}$$

$$+\left(1-\frac{1}{\mu_b}\right)\sum_{j=0}^{\infty}\left(\frac{1}{\mu_b}\right)^j\left\{-r_{t+j-1}+\phi\frac{\mu_h}{\mu_b}\pi_{t+j}+\left(1+\phi\frac{\mu_h}{\mu_b}\right)\gamma_{t+j}\right\} \tag{FINAL}$$

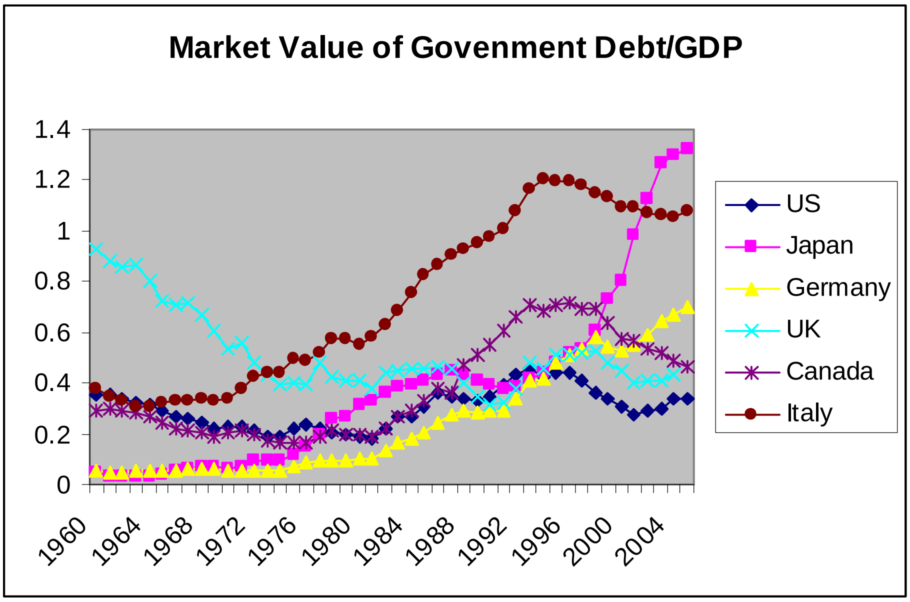

Figure 1a : Market Value of Government Debt to GDP

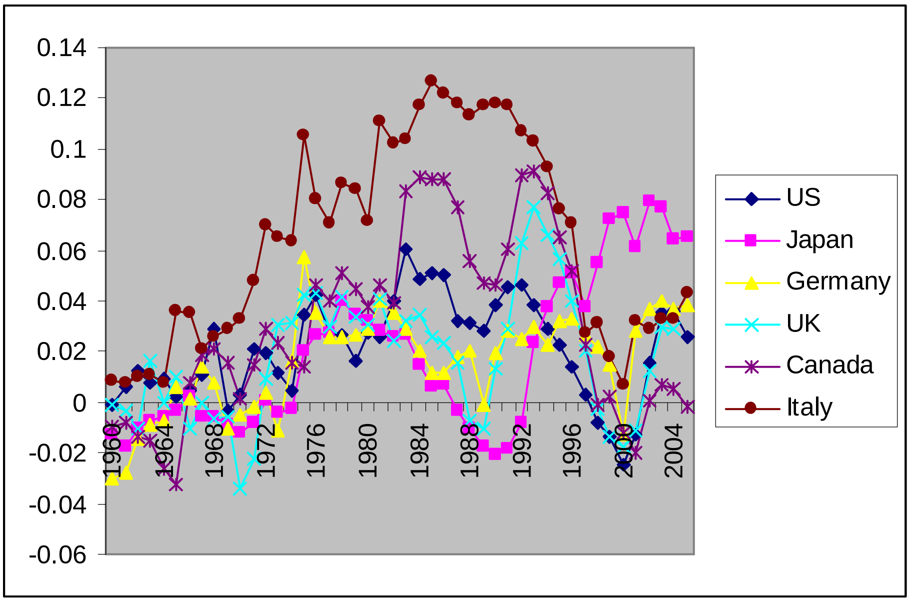

Figure 1b : Total Deficit/GDP

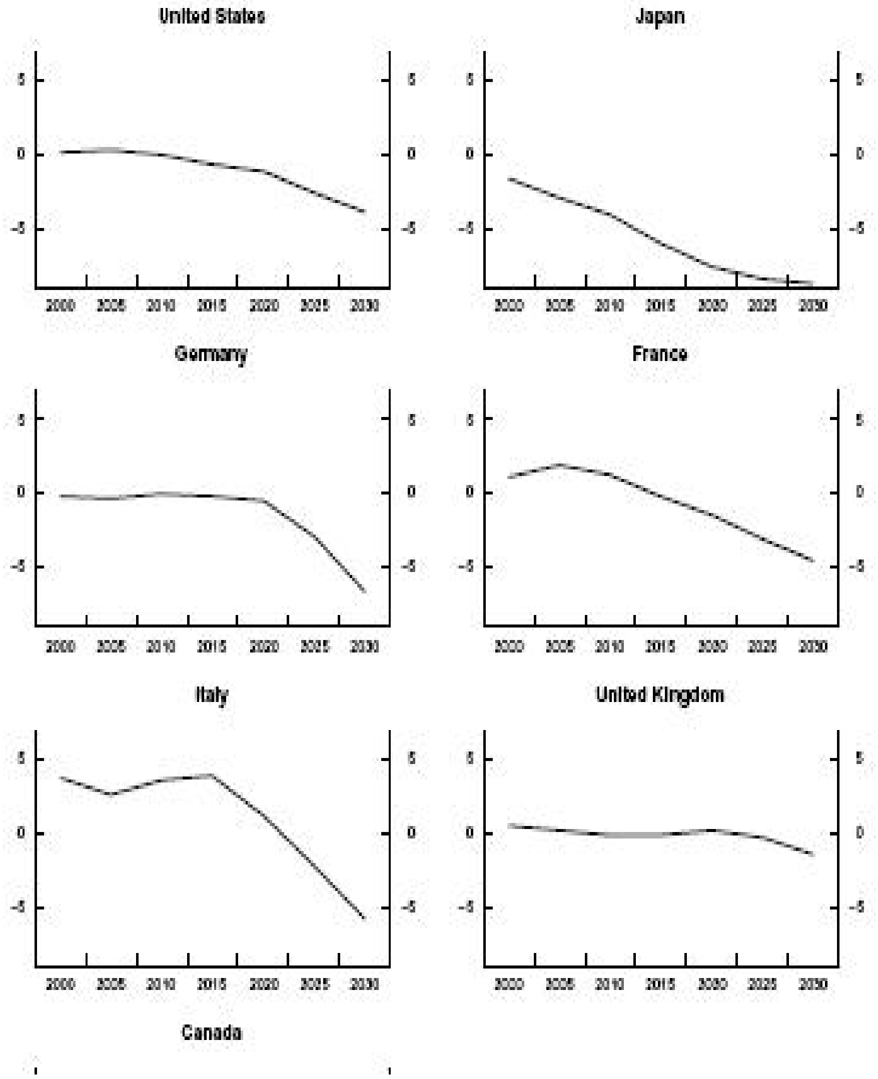

Figure 2 : Projected Primary Deficits 2000-2030
Source : ***Ageing Populations, Pension Systems and Government Budgets***
Roseveare, Leibfritz, Fore and Wurzel OECD Economics Department Working Paper 168, 1998

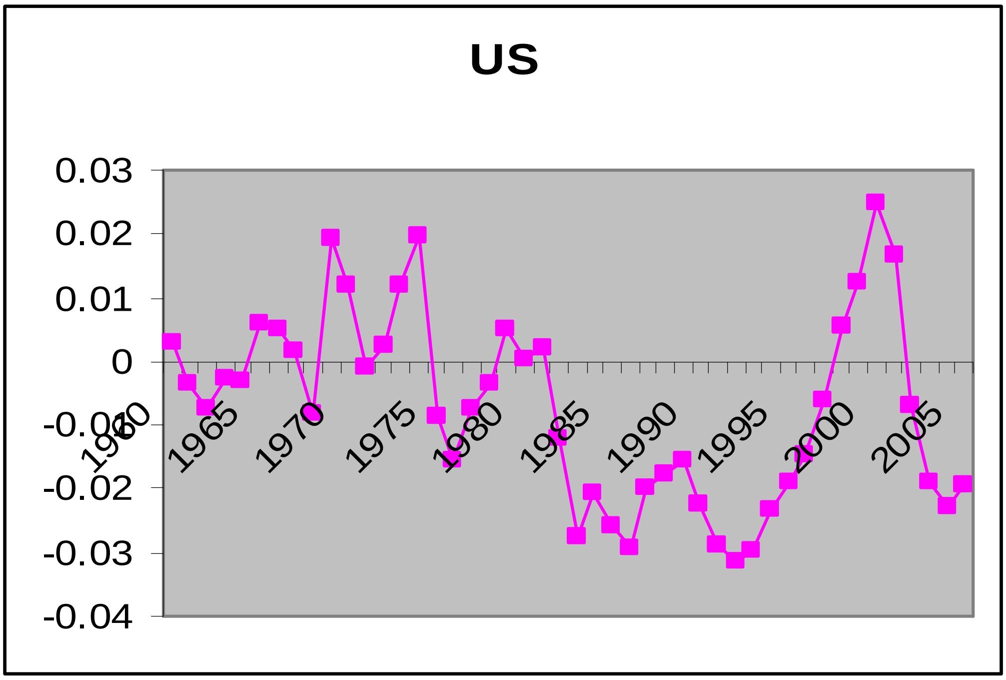

Figure 3a – Fiscal Position US

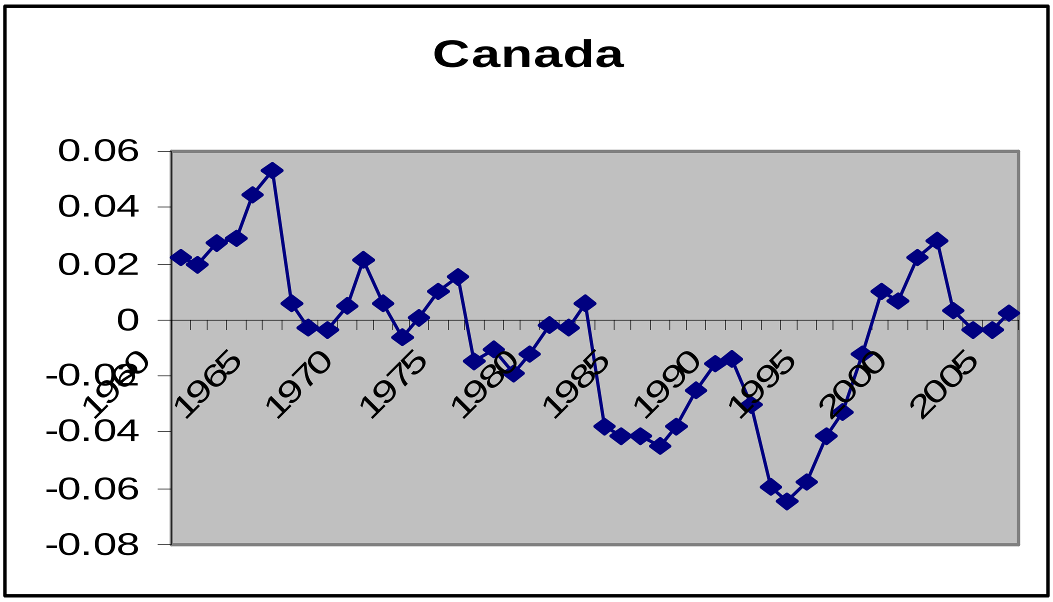

Figure 3b – Fiscal Position Canada

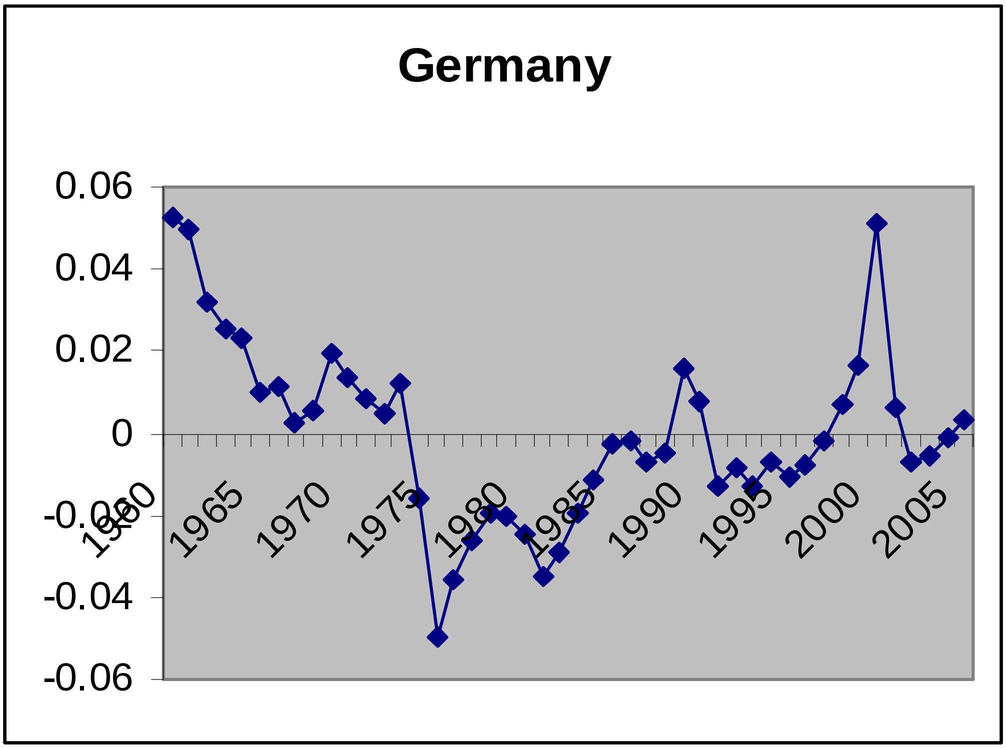

Figure 3c – Fiscal Position Germany

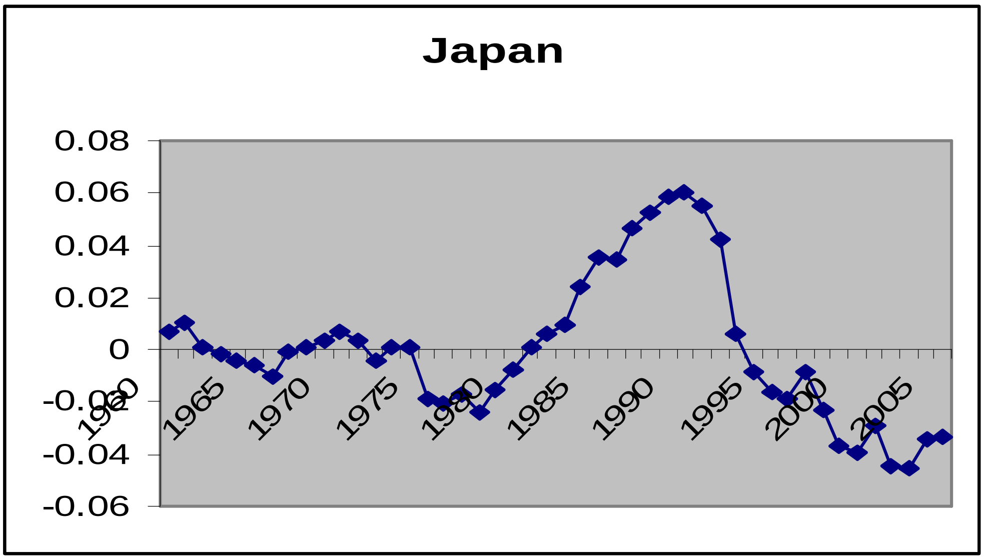

Figure 3d – Fiscal Position Japan

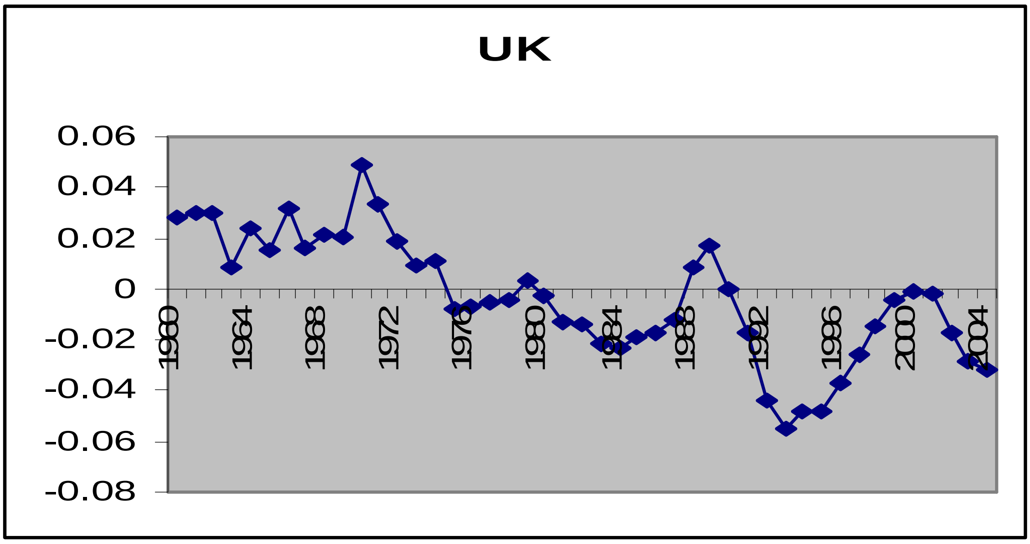

Figure 3e – Fiscal Position UK

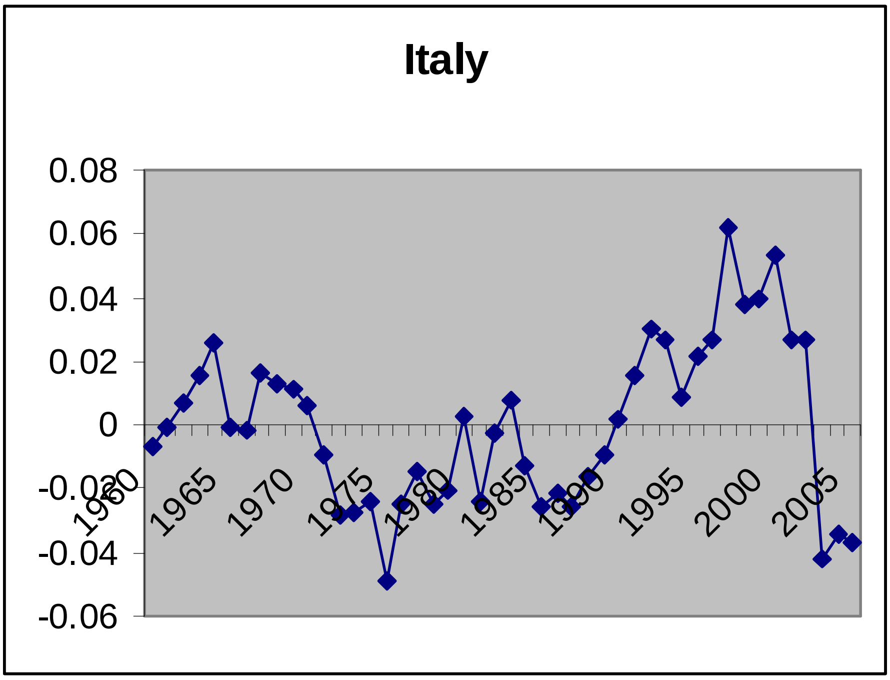

Figure 3f – Fiscal Position Italy

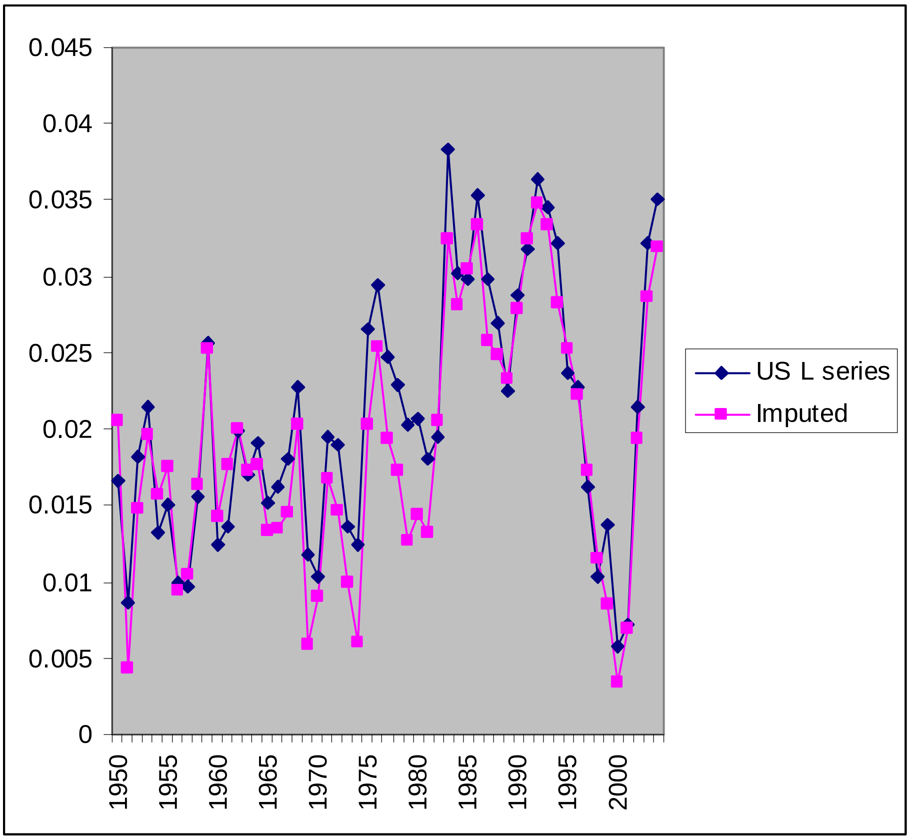

Figure 4 – US Decomposition

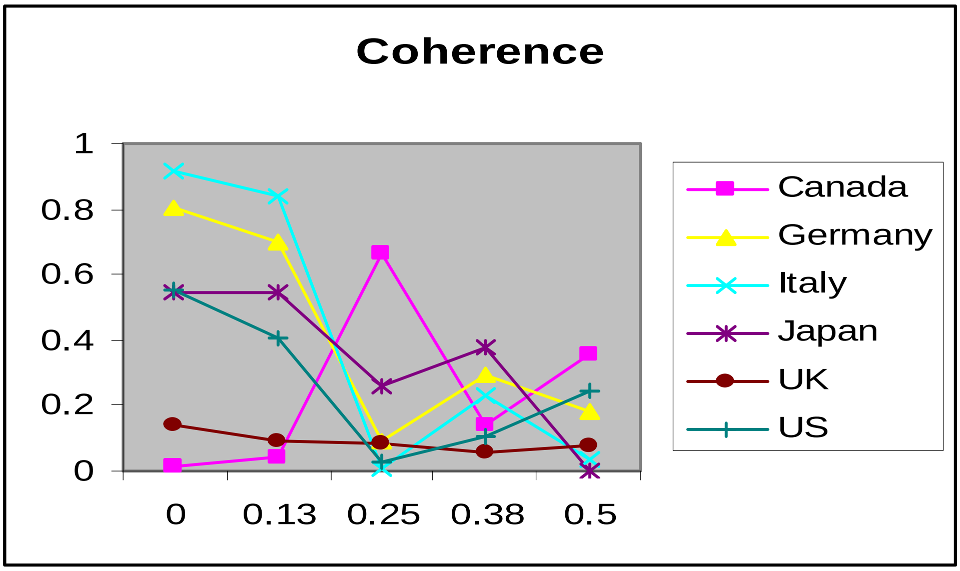

Figure 5 – Coherence Estimates