Bubbles and Panics in a Frictionless Market with Heterogeneous Expectations

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Abstract

When investors have differences of opinion about the payoffs of a stock, Harrison and Kreps (1978) demonstrate the existence of a speculative bubble in the stock price, that is, the stock price can exceed the valuation of the most optimistic investor. A crucial condition that supports this result in their model is that investors are not allowed to short sell the stock. This paper demonstrates that speculative bubbles may arise even without the short sales constraint. The paper also demonstrates that asset panics may arise, that is, the stock price may be lower than the valuations of all individual investors. In particular, even if the short sales constraint binds, asset panics can still arise. This result suggests that Miller's (1977) insight that the short sales constraint causes the stock price to be above the average valuation is not robust in a dynamic framework. In the case of a bubble, our model generalizes the Harrison-Kreps notion of a resale option, namely, investors believe that they can resell the stock later at a higher price. In the case of a panic, our model develops the notion of a buy-back option, namely, investors believe that they can sell the stock now and buy back the stock later at a lower price. Intuitive sufficient conditions for bubbles and panics are developed, and it is shown that under certain conditions, bubbles and panics can arise even when investors have almost homogeneous expectations.

1 Introduction

In recent years, the market has experienced the dramatic rise and fall of the internet stocks. For example, Ofek and Richardson (2003) report that in the two-year period from early 1998 through February 2000, the Internet sector earned over 1000 percent returns on its public equity, but these returns had completely disappeared by the end of 2000. Spiegel (2004) reviews extensively the stock market performance and offers a comprehensive discussion on bubbles and panics.

In a survey study, Dhar and Goetzmann (2005) find that many investors admitted to buying stocks they believed at the time to be overvalued, but claimed to have done so on the anticipation that the share prices would continue to rise. See also Vissing-Jorgensen (2004). These findings seem to be difficult to reconcile with the asset pricing models in which all investors share the same beliefs about asset returns. They provide potential evidence that investors have heterogeneous expectations about the stock payoffs.

In a seminal paper in which risk-neutral investors have differences of opinion about the asset payoffs as well as there are short sales constraints, Harrison and Kreps (1978) obtain a remarkable result that the price of an asset can be higher than the valuation of even the most optimistic investor in the market.[1] The intuition is that investors price an asset based not only on the payoffs associated with the ownership of the asset but also on the right to resell the asset. With the short sales constraint and risk-neutral investors, the asset price in every future state is determined by the investor who is most optimistic about the asset payoffs in that state. With differences of opinion, an investor may have the highest valuation today on expected basis, but he may not have the highest valuation in all future states. As a result, the stock price today may exceed the valuation of the most optimistic investor (based on the expected future prices under the investor's beliefs) or speculative bubbles may occur.[2]

[1] In this paper we define short sales constraints as any transactions costs or restrictions associated with short selling an asset. In the original Harrison-Kreps model and most of its extensions, investors are simply not permitted to short sell the asset.

[2] Harris and Raviv (1993), Kandel and Pearson (1995), Morris (1996), Biais and Bossaerts (1998), Duffie, Garleanu, and Pederson (2002), Kyle and Lin (2002), Viswanathan (2002), Hong and Stein (2003), Scheinkman and Xiong (2003), and Cao and Ou-Yang (2004) have extended the Harrison-Kreps model in various aspects. For comprehensive

In addition, Miller (1977) argues in a one-period arrangement that the short sales constraint makes it difficult for pessimistic investors to participate in the stock market and that even if these investors believe that the stock price is overvalued, they have no means to bring the price down. As a result, the stock price is determined by the average of the more optimistic investors' valuations.

Due to the risk neutrality of investors and the short sales constraint in Harrison and Kreps (1978), panics will never occur or the stock price will never be lower than the valuation of the most pessimistic investor. In this model, the stock price in every state of nature is determined by the investor who has the highest valuation in that state because the short sales constraint binds for all but the most optimistic investor about the state. Due to the static nature of Miller (1977), neither bubbles nor panics can occur because the stock price is determined by the weighted average valuation of the investors who participate in the stock market and the weighed average valuation cannot be higher than that of the most optimistic investor or lower than that of the most pessimistic investor.

Key conditions that lead to speculative bubbles in the Harrison-Kreps model as well as in its extensions are the presence of the short sales constraint and the risk neutrality of investors. This paper shows that neither the short sales constraint nor the risk neutrality is necessary for asset bubbles to occur. When investors are risk averse, we demonstrate that both bubbles and panics can occur in the absence of the short sales constraint. Somewhat strikingly, we find that even when the short sales constraint binds, the stock price may still be lower than all investors' valuations. This result suggests that the short sales constraint is neither necessary nor sufficient for bubbles and panics to arise.

We consider two models with risk-averse investors. In the first model, we impose no restrictions on stock payoffs and assume that investors are myopic and maximize the expected utility of the next-period wealth. There are three periods, 0, 1, and 2. A publicly observed signal, which is correlated with the stock payoff, arrives in period 1. An asset bubble is

reviews, see, for example, Rubinstein (2004) and Scheinkman and Xiong (2004). Asset bubbles may also arise in other settings with certain restrictions, such as those of De Long et al. (1990), Allen and Gorton (1993), Allen, Morris, and Postlewaite (1993), and Abreu and Brunnermeier (2003), and DeMarzo, Kaniel, and Kremer (2004).

said to arise when the stock price at time 0 is higher than the valuation based on the most optimistic belief. Similarly, a panic is said to arise when the stock price at time 0 is lower than the valuation based on the most pessimistic belief. To determine the stock price in the current period, investors must consider the next-period stock prices or the resale values of the stock.

We show that under certain conditions, the stock price today can be higher than the valuation of the most optimistic investor, and under other conditions, the price can be lower than the valuation of the most pessimistic investor. When investors are risk averse, the stock price today is equal to the average of all investors' expectations of the next-period prices, adjusted for the investors' conditional precisions,[3] minus a term for the risk premium. With differences of opinion about the stock payoff, it is possible that this weighted average of all investors' expectations can be greater than the highest unconditional expectation of the final stock payoffs, or lower than the lowest unconditional expectation of the final stock payoffs.

For example, suppose that there are two investors, i and j, and that there are two states, A and B at time 1. Investor i believes that there is a 1/4 probability that A occurs with a conditional expected payoff of 3 and that there is a 3/4 probability that B occurs with a conditional expected payoff of 2. Investor i's expectation of the final payoff is then given by 2.25. Investor j believes that the two states will occur with an equal probability and that the conditional expected payoffs in A and B are 2.2 and 2, respectively. The expected final payoff of investor j is then given by 2.1. Hence, investor i has the higher valuation. Note that investor i expects the payoff to be 3 in state A or that he is very optimistic in this state. When investor j uses his own beliefs to calculate the expected value of the payoffs believed by investor i, investor j arrives at (3/4)*3+(1/4)*1=2.5, which is higher than both his own valuation of 2.1 and investor i's valuation of 2.25. The reason is that investor j believes that state A will occur with a higher probability and that investor i is very optimistic in this state. It is then possible that the stock price today is higher than the valuation of investor i,

[3]For example, if an investor's precision of a signal is high, then he will demand a lower risk premium for the stock, which affects the stock price.

because investor j may believe that he can resell the stock to investor i at a higher price in the next period when state A occurs. Consider an extreme case in which investor i has the same beliefs as in the previous case but investor j believes that with a probability of 1 state A will occur at time 1 and that his expectation of the stock payoffs in this state is 2.1. In this case investor i is still more optimistic than investor j, but investor j may well be willing to pay more than 2.25 for the stock today because he believes that at time 1 state A will be realized and that investor i's expected value is 3. In other words, investor j believes that he can resell the stock to investor i at an even higher price.

In an economy with two types of investors and two possible states (A and B), we find that a sufficient condition for bubbles to arise is that the less optimistic type of investors expects one of the states to happen at time 1 with a sufficiently higher probability than the more optimistic type, who has a higher valuation in this state. It can be the case that both types of investors have high valuations in that state and low valuations in the other state or that they agree on the direction of the signals. For example, when state A is realized, both types of investors expect the stock payoffs to be high and when state B is realized, both types expect the stock payoffs to be low, but they differ on the magnitudes.[4] Intuitively, the less optimistic investors are willing to pay a high price for the stock today because they expect to resell the stock later at an even higher price to more optimistic investors.

Similarly, we find that panics can arise when the more optimistic investors believe that one of the states is very likely to occur at time 1 and that the less optimistic investors will value the stock very low in that state. As a result, the more optimistic investors are willing to sell the stock today at a low price because they expect to buy back the stock later at an even lower price from the less optimistic investors. In particular, even if the short sales constraint binds for some of the investors, asset panics can still arise. When the short sales constraint prevents some investors from trading, the rest of the investors must hold more shares of the risky stock. They demand a high risk premium so that the stock price can be

[4]The results become stronger when different types of investors have high and low valuations in different states or when they disagree on the direction of the signals.

very low, as discussed in Subsection 3.1.

In sum, in the case of asset bubbles, our model generalizes the Harrison-Kreps notion of a resale option to an economy with risk-averse investors but without the short sales constraint. Some investors may purchase the stock today at a high price because they believe that the stock price will be even higher in the future. This captures the sprit of the survey results of Dhar and Goetzmann (2005), who find that investors buy overvalued stocks because they anticipate the share prices would continue to rise. In the case of asset panics, our model develops a buy-back option for risk-averse investors. If investors expect the stock price to be low in the future, then they may be willing to sell the stock at a low price today hoping to buy back the stock later at an even lower price. We further demonstrate that panics may occur even when the short sales constraint binds in the current trading period and is allowed in future trading periods.

One may argue that asset bubbles and panics arise because investors have myopic views in the model. We then show that the results still go through when investors are forward looking in a dynamic model. As in the myopic model, there are three time periods in this economy, 0, 1, and 2. Investors trade in periods 0 and 1. Consumption occurs in period 2. To obtain closed-form solutions, we assume that investors have negative exponential utility functions and stock payoffs are normally distributed. We derive conditions under which asset bubbles or panics occur without the short sales constraint. Again, bubbles and panics occur due to the resale and buy-back options, respectively. Furthermore, we find that bubbles and panics may arise even if investors are almost homogeneous such that the differences in their interpretations of information are very small.

Empirically, investors are certainly risk averse with respect to the market-wide risk because there is strong evidence that the market risk premium is significantly positive. In addition, there is no compelling, direct evidence showing that the short sale constraint is necessary for asset bubbles to form. Reed (2001) and D'Avolio (2002) report that stocks are inexpensive to short in general. Lamont and Thaler (2003) notice that although internet stocks had higher average short interest and were more expensive to short than non-Internet stocks, the average difference in shorting costs was only 1% per year between 1998 and 2000. Jones and Lamont (2002) show that not only are the stocks in their sample overpriced, the magnitude of overpricing cannot be explained by measured shorting costs alone. In addition, Figlewski and Webb (1993) observe that the average short interest is only 0.2% of the outstanding shares for individual stocks. Brunnermeier and Nagel (2004) point out that short sales constraints are not sufficient to explain the failure of rational activity to contain the technology bubble. Lamont and Stein (2004) examine short interest for stock indices and find that total short interest moves in a counter cyclical fashion, that is, short interest actually declines as indices climb, and that short-selling does not particularly help stabilize the overall market. This result is striking because shorting indices is inexpensive and investors do not bear firm-specific risks for trading in indices. These studies suggest that the short sales constraint is perhaps not the main reason that asset bubbles occur.

In short, it appears to be empirically unrealistic to consider bubbles and panics with risk-neutral investors and the short sales constraint. This paper incorporates risk-averse investors as well as relaxes the short sales constraint in an integrated framework in which both bubbles and panics are discussed.

The rest of this paper is organized as follows. When investors are myopic, Section 2 shows that with differences of opinion alone, the asset price in the current period can exceed or be lower than the valuations of all investors. Section 3 compares our results with those of Harrison and Kreps (1978) and Miller (1977). When investors have negative exponential utility functions and maximize their expected utilities through dynamic trading, Section 4 demonstrates that both bubbles and panics can still arise without the short sales constraint. Section 5 concludes the paper. The appendix contains technical proofs.

2 A Myopic Mean-Variance Model

For the simplicity of exposition, we consider a three-period model, with a time line of 0, 1, and 2. There is one risk free bond and one risky stock available for trading. It is assumed that the financial market is populated by investors with the population size normalized to one, each indexed by i where $i \in [0, 1]$. Trading takes place because investors have differences of opinion about the terminal payoff (at time 2) of the stock. At time 0, we assume that each investor is endowed with x^i units of the stock and zero units of the bond. Without loss of generality, the interest rate is taken to be zero. The stock payoff at time 2 is v. The per capita supply of the stock is a positive number denoted by x. We assume in this section that investors are myopic, that is, they maximize the expected utility period by period without considering the effects of future periods.

At time 1, a public signal y, which reveals information about the final payoff of the stock v, is made available to investors. Investors have different interpretations of y. For example, they may use different statistical techniques to learn about v through y. We assume that in each period investor i has mean-variance utility function given by

$$U_{ti} = \mathrm{E}_{ti}[W_{(t+1)i}] - \frac{1}{2}\gamma \mathrm{Var}_{ti}[W_{(t+1)i}], \quad t = 0, 1, \tag{1}$$

where γ denotes the investor's risk aversion coefficient and $W_{(t+1)i}$ denotes investor i's wealth at time $(t + 1)$.

At time 2, investor i's wealth is given by

$$W_{2i} = W_{1i} + D_{1i}(v - P_1). \tag{2}$$

Here W_{1i} denotes investor i's wealth at time 1 and the second term denotes his profit from investing in the stock, where D_{1i} is the investor's demand for the stock.

From Equation (1), we have that investor i's expected utility at time 1 is given by

$$U_{1i} = W_{1i} + D_{1i}(\mu_{1i} - P_1) - \frac{D_{1i}^2}{2}\gamma\sigma_{1i}^2, \quad t = 0, 1, \tag{3}$$

where $\mu_{1i} \equiv E_{1i}[v]$ denotes investor i's conditional expectation of v at time 1 and $\sigma_{1i}^2 \equiv \text{Var}_{1i}[v]$ denotes investor i's conditional variance of v at time 1. The first-order condition with respect to D_{1i} is given by

$$\mu_{1i} - P_1 - D_{1i}\gamma\sigma_{1i}^2 = 0. \tag{4}$$

The optimal demand at time 1 is then given by

$$D_{1i} = \frac{\mu_{1i} - P_1}{\gamma\sigma_{1i}^2}, \tag{5}$$

Let $\pi_{1i} \equiv 1/\sigma_{1i}^2$. Using the market clearing condition, $\int_i D_{1i}di = x$, we can express the equilibrium price P_1 as

$$P_1 = \mu_1 - \frac{\gamma x}{\pi_1}, \quad \pi_1 = \int_i \pi_{1i} di, \quad \mu_1 = \frac{1}{\pi_1}\int_i \pi_{1i}\mu_{1i} di, \tag{6}$$

where π_1 is the average precision over the entire population of investors and μ_1 is the precision weighted population average expectation of the stock payoff.

At time 1, investor i's wealth is given by

$$W_{1i} = W_{0i} + D_{0i}(P_1 - P_0), \tag{7}$$

where P_0 is the stock price at time 0 (today). Although investors are myopic, their maximization problems are connected through P_1. Similarly, we obtain the investor's demand for the risky stock and the current stock price:

$$D_{0i} = \frac{1}{\gamma}\pi_{0i}[P_1]\left(\text{E}_{0i}[P_1] - P_0\right), \tag{8}$$

$$P_0 = \mu_0[P_1] - \frac{\gamma x}{\pi_0[P_1]}, \tag{9}$$

where $\mu_0[P_1]$ is the precision weighted population average of the expected prices at time 1 (P_1) and $\pi_0[P_1]$ is the population average precision of P_1, $\text{E}_{0i}[P_1]$ represents investor i's conditional expectation of P_1 and $\pi_{0i}[P_1]$ is investor i's conditional precision of P_1, all of which are evaluated at time 0. The expressions for $\mu_0[P_1]$ and $\pi_0[P_1]$ are given by

$$\mu_0[P_1] = \frac{\int \mathrm{E}_{0i}[P_1]\pi_{0i}[P_1]di}{\int \pi_{0i}[P_1]di}, \tag{10}$$

$$\pi_0[P_1] = \int \pi_{0i}[P_1]di. \tag{11}$$

Let P_{ti} denote the stock price at time t if all investors share the same belief of type i. We have the following definitions of bubbles and panics.

Definition 1 *A bubble occurs at time t when the stock price (under heterogeneous beliefs) is higher than the highest stock price that would obtain if all investors were homogeneous. That is*

$$P_t > \max_i \{P_{ti}\}. \tag{12}$$

A panic occurs at time t when the stock price (under heterogeneous beliefs) is lower than the lowest stock price that would obtain if all investors were homogeneous. That is

$$P_t < \min_i \{P_{ti}\}.$$

Let $\pi_{0i}[P_1]$ denote investor i's precision of P_1 and $\pi_0[P_1] \equiv \int_i \pi_{0i}[P_1]di$ denote the population average precision of P_1. We have the expression for the precision weighted population average of expectations

$$\mu_0(P_1) = \frac{\int_i \pi_{0i}[P_1]\mathrm{E}_{0i}[\mu_1]di}{\int_i \pi_{0i}[P_1]di}. \tag{13}$$

We can then rewrite the price at time 0 as

$$P_0 = \mu_0(P_1) - \gamma x \left[\frac{1}{\pi_1} + \frac{1}{\pi_0(P_1)}\right].$$

For a stock bubble to occur at time 0, it is both necessary and sufficient to have

$$\mu_0(P_1) - \gamma x \left[\frac{1}{\pi_1} + \frac{1}{\pi_0(P_1)}\right] > \max_i \{P_{0i}\}.$$

Similarly, for a stock panics to occur at time 0, it is both necessary and sufficient to have

$$\mu_0(P_1) - \gamma x \left[\frac{1}{\pi_1} + \frac{1}{\pi_0(P_1)}\right] < \min_i\{P_{0i}\}.$$

For the simplicity of illustration, we next consider a case in which γx is close to zero or investors are close to risk neutral. In this case, $P_1 = \mu_1$. The condition for a bubble to occur further reduces to

$$\mu_0(\mu_1) > \max_i\{\mu_{0i}\} = \max_i\{\mathrm{E}_{0i}[\mathrm{E}_{1i}[v]]\} = \max_i\{\mathrm{E}_{0i}[v]\}. \tag{14}$$

When investors have homogeneous expectations, this condition cannot be satisfied because no investor's expectation exceeds the maximum expectation. When investors have heterogeneous expectations, we next discuss the necessary conditions for this inequality to hold.

Suppose that

- (1) the conditional precisions are deterministic for each investor and
- (2) $\mathrm{E}_{0i}[\mu_{1j}] = \mathrm{E}_{0i}[\mathrm{E}_{1j}[v]] = \mathrm{E}_{0i}[v]$ for all i, j.[5]

We then have

$$\mathrm{E}_{0i}[\mu_1] = \mathrm{E}_{0i}\left[\frac{\int_j \pi_{1j}\mu_{1j}dj}{\int_j \pi_{1j}dj}\right] = \frac{\int_j \pi_{1j}\mathrm{E}_{0i}[v]dj}{\int_j \pi_{1j}dj} = \mathrm{E}_{0i}[v] \leq \max_i\{\mathrm{E}_{0i}[v]\}.$$

As a result, we have that

$$P_0 = \mu_0(\mu_1) \leq \max_i\{\mathrm{E}_{0i}[v]\},$$

which follows from the fact that the weighted average of certain values does not exceed the maximum value. Therefore, for an asset bubble to occur, one of the two conditions must fail. Interestingly, when conditions (1) and (2) are satisfied, bubbles may still occur in a Harrison-Kreps setting with risk neutral investors under short sale contraint.

[5]The precisions do not have to be equal across investors. In a noisy rational expectations equilibrium, Allen, Morris, and Shin (AMS, 2004) are perhaps the first to demonstrate that the law of iterated expectations may not hold for the average expectations. In their model, the demand for the stock is caused by an exogenous supply shock, whereas in our model, the demand for the stock is due to the assumption that investors have heterogeneous beliefs. AMS do not consider the issues discussed in the current paper, that is, the stock price can either exceed the valuation of the most optimistic investor or drop below the valuation of the most pessimistic investor.

Suppose that investor i's expectation is the highest among all investors. For a bubble to occur, if condition (1) holds, then it is necessary that $\mathrm{E}_{0i}[\mathrm{E}_{1j}[v]] > \mathrm{E}_{0i}[v]$ for some i such that investor i values investor j's expected payoffs more highly than his own payoffs. In other words, investor i expects investor j to value the stock more highly in the next period. Otherwise, investor i would not pay more than his own valuation for the stock at time 0.

Even if condition (2) is satisfied or $\mathrm{E}_{0i}[\mathrm{E}_{1j}[v]] = \mathrm{E}_{0i}[v]$, it is still possible that

$$\mathrm{E}_{0i}[\mu_1] = \mathrm{E}_{0i}\left[\frac{\int_j \pi_{1j}\mu_{1j}dj}{\int_j \pi_{1j}dj}\right] > \max_i\{\mathrm{E}_{0i}[v]\},$$

where π_{1j} must be stochastic. Intuitively, under investor i's probability belief, investor j has high precisions (π_{1j}) in the high payoff states (μ_{1j}) or investor j is willing to pay a high price in those states. Essentially, investor i expects other investors to value the stock more highly than he does. In other words, an investor is willing to buy the stock at a high price today because he expects to resell the stock at an even higher price later.

In sum, for an asset bubble to occur, even if an investor has the highest valuation of a stock under homogeneous beliefs, he expects other investors to value the stock even higher under heterogeneous beliefs. On average, investors are willing to pay a high price today because they believe that they can resell the stock at an even higher price. Similarly, for an asset panic to occur, investors must expect that the stock price in the next period will be even lower, so they would rather wait to buy the stock later. In equilibrium, the stock price today must be low to clear the market. Our analysis indicates that a bubble or a panic can only occur in a dynamic setting in which a resale option or a buy-back option exists. We next provide numerical examples for bubbles and panics as well as develop sufficient conditions in a two-state, two-investor economy.

2.1 Numerical Examples

Example 1: A Bubble

Suppose that there are two types of investors, type i and type j, with equal proportion.

The risk free rate is taken to be zero without loss of generality. Type i investors believe that the final stock payoff is 10 in state H and 0 in state L with equal probability. Type j investors believe that the final stock payoff is 10 in state H with a probability of 0.49 and 0 in state L with a probability of 0.51. The unconditional expectations are 5 and 4.9 for types i and j, respectively, so under homogeneous beliefs, type i is more optimistic than type j. We next demonstrate that the equilibrium stock price at time 0 can be higher than 5.

At time 1, a public signal arrives with two possible realizations, A and B. For investors of type i, the probability of realization A is $\text{Prob}^i(A) = 1/4$ and the probability of realization B is $\text{Prob}^i(B) = 3/4$. When the signal realization is A, the conditional probability of state H is $4/5$. When the signal realization is B, the conditional probability of state H is $2/5$. For investors of type j, the probability of state A is 0.45 and the probability of state B is 0.55, that is, $\text{Prob}^j(A) = 0.45$ and $\text{Prob}^j(B) = 0.55$. When the signal realization is A, the conditional probability of state H is 0.6. When the signal realization is B, the conditional probability of state H is 0.4. Let $\mu_{1a}(s)$ and $\pi_{1a}(s)$ denote, respectively, the conditional mean and conditional precision of investor type a ($a = i, j$) and with signal realization s ($s = A, B$) at time 1. Notice that investors j believe that realization A is more likely to occur at time 1.

Given the above data, we can calculate the investors' conditional expectations and precisions. Specifically, we obtain that

$$\mu_{1i}(A) = 8, \quad \mu_{1i}(B) = 4, \quad \mu_{1j}(A) = 6, \quad \mu_{1j}(B) = 4,$$

$$\pi_{1i}(A) = \frac{1}{16}, \quad \pi_{1i}(B) = \frac{1}{24}, \quad \pi_{1j}(A) = \frac{1}{24}, \quad \pi_{1j}(B) = \frac{1}{24}.$$

It can be checked that the unconditional expectations of the final payoffs are 5 and 4.9 for investors i and j, respectively.

Assume that all investors are close to risk neutral so that we can ignore the risk premium term in the stock price. When the signal is A at time 1, the stock price is then approximately given by

$$
\begin{aligned}
P_1(A) &= [\pi_{1i}(A) + \pi_{1j}(A)]^{-1} \times [\pi_{1i}(A)\mu_{1i}(A) + \pi_{1j}(A)\mu_{1j}(A)] \\
&= \left[\frac{1}{16} + \frac{1}{24}\right]^{-1} \times \left[\frac{8}{16} + \frac{6}{24}\right] = 7.2.
\end{aligned}
$$

When the signal is B, the stock price is given by

$$
\begin{aligned}
P_1(B) &= [\pi_{1i}(B) + \pi_{1j}(B)]^{-1} \times [\pi_{1i}(B)\mu_{1i}(B) + \pi_{1j}(B)\mu_{1j}(B)] \\
&= \left[\frac{1}{24} + \frac{1}{24}\right]^{-1} \times \left[\frac{4}{24} + \frac{4}{24}\right] = 4.
\end{aligned}
$$

The stock price at time 0 is determined by the average of the conditional expectations of the two investors, adjusted by their conditional precisions. The investors' conditional expectations are

$$
\mathrm{E}_{0i}[P_1] = \frac{1}{4} \times 7.2 + \frac{3}{4} \times 4 = 4.8,
$$

$$
\mathrm{E}_{0j}[P_1] = 0.45 \times 7.2 + 0.55 \times 4 = 5.44.
$$

Their conditional precisions are

$$
\pi_{0i}[P_1] = \left[\frac{1}{4}(7.2 - 4.8)^2 + \frac{3}{4}(4.4 - 4)^2\right]^{-1} = 0.5208,
$$

$$
\pi_{0j}[P_1] = \left[0.45 \times (7.2 - 5.44)^2 + 0.55 \times (5.44 - 4)^2\right]^{-1} = 0.3946.
$$

The stock price at time 0 is then given by

$$
\begin{aligned}
P_0 &= [\pi_{0i}[P_1] + \pi_{0j}[P_1]]^{-1} [\pi_{0i}[P_1]\mathrm{E}_{0i}(P_1) + \pi_{0j}[P_1]\mathrm{E}_{0j}(P_1)] \\
&= (0.5208 + 0.3946)^{-1} \times (0.5208 \times 4.8 + 0.3946 \times 5.44) = 5.08 > 5.
\end{aligned}
$$

Type j investors are willing to acquire the stock at 5.08 at time 0 because their expected time 1 price is 5.44. Type i investors sell the stock because their expected time 1 price is only 4.8. Although the unconditional expected payoff of type i investors is 5, the expected payoff of type j, based on type i's probability beliefs, is given by $1/4 \times 6 + 3/4 \times 4 = 4.5$. This means that investor i believes that he can buy the stock at a low price in the next period, so he sells the stock today. On the other hand, the expected payoff of type i based on type j's probability beliefs, is given by $0.45 \times 8 + 0.55 \times 4 = 5.8$. It means that investor j believes that he can resell the stock to investor i at a higher price in the next period, so he buys the stock today at a high price that exceeds the valuation of investor i. More specifically, if signal A is realized at time 1, both types of investors, type i in particular, are very optimistic about the stock payoffs, and if signal B is realized at time 1, both types of investors are relatively pessimistic about the stock payoffs. At time 0, type i investors believe that signal A is more likely to occur whereas type j investors believe that signal B is more likely to occur. As a result, type j investors may be willing to pay a high price for the stock today because they hope to sell the stock to type i investors at time 1, and type i investors are happy to sell the stock today at a high price because they expect to buy back the stock at a lower price at time 1. In other words, a bubble arises due to the perceived resale option of investor j.

Example 2: A Panic

There are two types of investors, type i and type j, with equal proportion. Type i investors believe that the stock payoff is 10 or 0 with equal probability. Type j investors believe that the stock payoff is 10 with a probability of 0.51 and 0 with a probability of 0.49. The unconditional expectations are given by 5 and 5.1 for types i and j, respectively, so type i is more pessimistic than type j. We next demonstrate that the equilibrium stock price at time 0 can be lower than 5.

At time 1 there is a signal with two possible realizations A and B. For i, A occurs with 1/4 probability. Conditional on A, 0 occurs with 4/5, and conditional on B, 0 occurs with 2/5. For j, A occurs with 0.45 probability. Conditional on A, 0 occurs with 0.6, and conditional on B, 0 occurs with 0.4. Notice that type j believes that realization A is more likely to occur.

Given the above data, we can calculate the investors' conditional expectations and precisions. We obtain that

$$\mu_{1i}(A) = 2, \quad \mu_{1i}(B) = 6, \quad \mu_{1j}(A) = 4, \quad \mu_{1j}(B) = 6,$$

$$\pi_{1i}(A) = \frac{1}{16}, \quad \pi_{1i}(B) = \frac{1}{24}, \quad \pi_{1j}(A) = \frac{1}{24}, \quad \pi_{1j}(B) = \frac{1}{24}.$$

It can be checked that the unconditional expectations of the final payoffs are 5 and 5.1 for investors i and j, respectively.

Assume that all investors are close to risk neutral so that we can ignore the risk premium term in the stock price. When the signal is A at time 1, the stock price is then approximately given by

$$\begin{aligned} P_1(A) &= [\pi_{1i}(A) + \pi_{1j}(A)]^{-1} \times [\pi_{1i}(A)\mu_{1i}(A) + \pi_{1j}(A)\mu_{1j}(A)] \\ &= \left[\frac{1}{16} + \frac{1}{24}\right]^{-1} \times \left[\frac{2}{16} + \frac{4}{24}\right] = 2.8. \end{aligned}$$

When the signal is B, the stock price is given by

$$\begin{aligned} P_1(B) &= [\pi_{1i}(B) + \pi_{1j}(B)]^{-1} \times [\pi_{1i}(B)\mu_{1i}(B) + \pi_{1j}(B)\mu_{1j}(B)] \\ &= \left[\frac{1}{24} + \frac{1}{24}\right]^{-1} \times \left[\frac{6}{24} + \frac{6}{24}\right] = 6. \end{aligned}$$

The price at time 0 is determined by the average of the conditional expectations of the two investors, adjusted by their conditional precisions. The investors' conditional expectations are given by

$$\mathrm{E}_{0i}[P_1] = \frac{1}{4} \times 2.8 + \frac{3}{4} \times 6 = 5.2,$$

$$\mathrm{E}_{0j}[P_1] = 0.45 \times 2.8 + 0.55 \times 6 = 4.56.$$

Their conditional precisions are

$$\pi_{0i}[P_1] = \left[\frac{1}{4}(5.2 - 2.8)^2 + \frac{3}{4}(6 - 5.2)^2\right]^{-1} = 0.5208,$$

$$\pi_{0j}[P_1] = \left[0.45 \times (4.56 - 2.8)^2 + 0.55 \times (6 - 4.56)^2\right]^{-1} = 0.3946.$$

The stock price at time 0 is then given by

$$\begin{aligned} P_0 &= [\pi_{0i}[P_1] + \pi_{0j}[P_1]]^{-1} [\pi_{0i}[P_1]\mathrm{E}_{0i}(P_1) + \pi_{0j}[P_1]\mathrm{E}_{0j}(P_1)] \\ &= (0.5208 - 0.3946)^{-1} \times (0.5208 \times 5.2 + 0.3946 \times 4.56) = 4.92 < 5. \end{aligned}$$

Type i investors buy the stock and type j investors sell the stock. This seems to be counterintuitive because in the absence of the other type of investors, types j investors are more optimistic than type i investors. Under heterogenous beliefs, however, the stock price depends not only on the investors' own expectations of the stock payoffs but also on the investors' expectations of other investors' believed payoffs. For example, under type i's probability beliefs, the expected value of type j's believed payoffs is given by $1/4 \times 4 + 3/4 \times 6 = 5.5$. Under type j's probability beliefs, the expected value of type i's payoffs is given by $0.45 \times 2 + 0.55 \times 6 = 4.2$. Although investors j value the stock at 5.1 in the absence of investors i, they expect the stock price to be much lower than 5 in the next period due to the presence of investors i. Investors j are willing to sell the stock today at a price below 5 because they believe that they can buy back the stock at an even lower price in the next period. This buy-back option causes the stock price to be lower than 5 or a panic arises. More specifically, investors j believe that realization A is more likely to occur at time 1 and that in A the stock price will be very low.

Example 3: A Bubble with Stochastic Volatility

In the previous two examples, the law of iterated expectations are violated, that is, $\mathrm{E}_{i0}[\mathrm{E}_{j1}[v]] \neq \mathrm{E}_{i0}[v]$. In this example, we show that when the conditional volatility is stochastic, a bubble can still occur even if the law of iterated expectations holds across investors.

There are two types of investors, type i and type j, with equal proportion. For simplicity of exposition, assume that for both types, the stock payoff is 10 with probability 5/8 or 0 with probability 3/8. The expected payoff is then given by 6.25 for both types of investors. We next demonstrate that the equilibrium stock price at time 0 can be higher than 6.25.

At time 1 there is a signal with two possible realizations A and B and both investors believe that A and B will occur with equal probability. For i, conditional on A, 0 occurs with probability 1/4, and conditional on B, 0 occurs with probability 1/2. For j, conditional on A, 0 occurs with probability 1/2 and conditional on B, 0 occurs with probability 1/4. Notice that investors agree on the probabilities of realizations A and B and as a result, the law of iterated expectations holds across investors or $E_{i0}[E_{j1}[v]] = E_{i0}[v]$ and $E_{j0}[E_{i1}[v]] = E_{j0}[v]$.

Given the above data, we can calculate the investors' conditional expectations and precisions. We obtain that

$$\mu_{1i}(A) = 7.5, \quad \mu_{1i}(B) = 5, \quad \mu_{1j}(A) = 5, \quad \mu_{1j}(B) = 7.5,$$

$$\pi_{1i}(A) = \frac{4}{75}, \quad \pi_{1i}(B) = \frac{1}{25}, \quad \pi_{1j}(A) = \frac{1}{25}, \quad \pi_{1j}(B) = \frac{4}{75}.$$

It can be checked that the unconditional expectations of the final payoffs are 6.25 for both types of investors.

Assume that all investors are close to risk neutral so that we can ignore the risk premium term in the stock price. When the signal is A at time 1, the stock price is then approximately given by

$$\begin{aligned} P_1(A) &= [\pi_{1i}(A) + \pi_{1j}(A)]^{-1} \times [\pi_{1i}(A)\mu_{1i}(A) + \pi_{1j}(A)\mu_{1j}(A)] \\ &= \left[\frac{4}{75} + \frac{1}{25}\right]^{-1} \times \left[\frac{4}{75} \times 7.5 + \frac{1}{25} \times 5\right] = 6.4129. \end{aligned}$$

When the signal is B, the stock price is given by

$$\begin{aligned} P_1(B) &= [\pi_{1i}(B) + \pi_{1j}(B)]^{-1} \times [\pi_{1i}(B)\mu_{1i}(B) + \pi_{1j}(B)\mu_{1j}(B)] \\ &= \left[\frac{1}{25} + \frac{4}{75}\right]^{-1} \times \left[\frac{1}{25} \times 5 + \frac{4}{75} \times 7.5\right] = 6.4129. \end{aligned}$$

The price at time 0 is determined by the average of the conditional expectations of the two investors, adjusted by their precisions. Since the prices at time 1 are the same in both states A and B, the stock price at time 0 is also 6.4129. Hence, the stock price with heterogeneous beliefs is larger than 6.25, the price that would obtain if all investors have the same beliefs as either type i or type j. In other words, a bubble can occur even if the law of iterated expectations holds across investors. Again, the investors' perceived resale option lead to the bubble.

2.2 Sufficient Conditions for Bubbles and Panics in a Two-State, Two-Investor Economy

We have examined examples for bubbles and panics to exist. The general sufficient conditions for the existence of bubbles and panics are very complex. To simplify the analysis, we consider only sufficient conditions for the existence of bubbles and panics in a two-state, two-investor economy. Let q_{aA} be the probability of state A for investor $a = i, j$. Let q_{aSH} be investor a's conditional probability that given signal $S = A, B$ the high state H occurs. Let the payoff in the high state be V_H and the payoff in the low state be V_L. For simplicity of exposition, assume that both types of investors have the same unconditional expectation, that is,

$$\begin{aligned} \mu &\equiv q_{iA}[q_{iAH}V_H + (1 - q_{iAH})V_L] + (1 - q_{iA})[q_{iBH}V_H + (1 - q_{iBH})V_L] \\ &= q_{jA}[q_{jAH}V_H + (1 - q_{jAH})V_L] + (1 - q_{jA})[q_{jBH}V_H + (1 - q_{jBH})V_L]. \end{aligned} \tag{15}$$

Given the signals at time 1, the conditional expectations of investors i and j are given by

$$\mu_{i1A} = q_{iAH}V_H + (1 - q_{iAH})V_L, \quad \mu_{i1B} = q_{iBH}V_H + (1 - q_{iBH})V_L,$$

$$\mu_{j1A} = q_{jAH}V_H + (1 - q_{jAH})V_L, \quad \mu_{j1B} = q_{jBH}V_H + (1 - q_{jBH})V_L.$$

Suppose that investors have very small risk aversion so that we can ignore the risk premium term in the price functions. Let π_{a1S} denote the conditional precision for investor a at time 1 given signal S. We have that the stock price at time 1 given signal S is given by the precision weighted average of the investors' conditional expectations of the stock payoffs:

$$P_{1S} = \frac{\pi_{i1S}\mu_{i1S} + \pi_{j1S}\mu_{j1S}}{\pi_{i1S} + \pi_{j1S}}.$$

Similarly, the stock price at time 0, P_0, is approximately given by the precision weighted average of the investors' expectations of the time 1 prices (P_{1S}):

$$P_0 = \left[\frac{\pi_{i0S}\mu_{0iS}(P_{1S}) + \pi_{j0S}\mu_{0jS}(P_{1S})}{\pi_i + \pi_j} \right],$$

where $\mu_{0a}(P_{1S})$ is investor a's precision weighted expectation of P_{1S} and π_a is investor a's conditional precision of P_{1S}.

For the stock price at time 0 to be larger than μ, it is sufficient to have

$$(\pi_i + \pi_j)^{-1} \left\{ \pi_i \left[\frac{q_{iA}\pi_{j1A}(\mu_{j1A} - \mu_{i1A})}{\pi_{i1A} + \pi_{j1A}} + \frac{(1 - q_{iA})\pi_{j1B}(\mu_{j1B} - \mu_{i1B})}{\pi_{i1B} + \pi_{j1B}} \right] \right.$$
$$\left. + \pi_j \left[\frac{q_{jA}\pi_{i1A}(\mu_{i1A} - \mu_{j1A})}{\pi_{i1A} + \pi_{j1A}} + \frac{(1 - q_{jA})\pi_{i1B}(\mu_{i1B} - \mu_{j1B})}{\pi_{i1B} + \pi_{j1B}} \right] \right\} > 0. \tag{16}$$

Notice that

$$\mu_{i1S} - \mu_{j1S} = (q_{iSH} - q_{jSH})(V_H - V_L).$$

We then have

$$(\pi_i q_{iA} \pi_{j1A} - \pi_j q_{jA} \pi_{i1A})(q_{jAH} - q_{iAH})(\pi_{i1B} + \pi_{j1B})$$
$$+ (\pi_i q_{iB} \pi_{j1B} - \pi_j q_{jB} \pi_{i1B})(q_{jBH} - q_{iBH})(\pi_{i1A} + \pi_{j1A}) > 0. \tag{17}$$

If $q_{jAH} - q_{iAH} = 0$ and $q_{jBH} - q_{iBH} = 0$, then there will be no bubbles. In this case, at time 1, the investors' conditional expectations of the final payoffs are the same. As a result, at time 0, investor i's expected value of investor j's time 1 payoffs is the same as the expected value of his own payoffs. Therefore, at time 0, the stock price or the precision weighted average of the investors' expectations of the time 1 prices will not exceed the investors' own valuations.[6]

For simplicity, suppose that $q_{jBH} = q_{iBH}$ or given realization B at time 1, the two investors' conditional expectations of the final payoffs are equal, from inequality (17), we

[6] In general, if the unconditional expectation of investor i is higher than that of investor j, then the stock price will not exceed the valuation of investor i.

must have the following inequality for a bubble to occur:

$$(\pi_i q_{iA} \pi_{j1A} - \pi_j q_{jA} \pi_{i1A})(q_{jAH} - q_{iAH}) > 0. \tag{18}$$

The sufficient conditions are then given by

$$q_{jHA} > q_{iAH}, \quad \pi_{j1A} > \pi_{i1A}, \quad \pi_i q_{iA} > \pi_j q_{jA}. \tag{19}$$

$q_{jAH} > q_{iAH}$ means that given realization A at time 1, investor j's valuation is higher than investor i's valuation. $\pi_{j1A} > \pi_{i1A}$ means that at time 1, investor j will trade more aggressively due to a higher conditional precision. $\pi_i q_{iA} > \pi_j q_{jA}$ means that investor i places more weight on realization A, driven by his precision at time 0 and his probability belief of realization A. Therefore, investor i believes that at time 1, investor j will be willing to pay a high price for the stock in state A. As a result, investor i is willing to pay a high price (higher than his own valuation) for the stock today.

Similarly, bubbles can arise under the following conditions:

$$q_{jAH} < q_{iAH}, \quad \pi_{j1A} < \pi_{i1A}, \quad \pi_i q_{iA} < \pi_j q_{jA}. \tag{20}$$

In other words, when A is realized at time 1, investor i's valuation is higher than investor j's valuation, but at time 0, investor j places more weight on realization A.

In sum, bubbles arise because one type of investors believes that the other type will be willing to pay a higher price for the stock in the future, or this type believes that it can resell the stock later at an even higher price. Specifically, one type of investors believes that a particular realization is likely to occur and that the other type values the stock very highly in that realization.

Similarly, for panics to occur, the more optimistic investors must believe that in a likely event, the stock price will drop a lot in the next period. As a result, these investors are willing to sell the today at a low price hoping to buy the stock back at an even lower price later. For example, type i places more weight on realization A at time 0, but this type believes that with realization A at time 1, type j will value the stock relatively low (lower than type i does). As a result, type i believes that he can buy back the stock from type j at a lower price at time 1, and it is possible that the stock price today is below the valuations of all investors.

3 Comparison with Harrison and Kreps (1978) and Miller (1977)

It is of interest to compare our result with that of Harrison and Kreps (1978) in which short sales constraints are essential and investors are risk neutral. In that model, the stock price at time 1 is given by

$$P_1 = \max_i \{\mathrm{E}_{1i}[v]\},$$

and the stock price at time 0 is given by

$$P_0 = \max_i \{\mathrm{E}_{0i}[P_1]\}.$$

Even with the short sales constraint, if one of the investors in the economy is always the most optimistic one in every state, then the stock price today will not exceed the valuation of this most optimistic investor or bubbles will not arise. In addition to the short sales constraint, the Harrison-Kreps model requires that for a bubble to occur, the most optimistic investor (based on the expected payoffs) must believe that in some of the future realizations, other investors will value the stock more highly than he does. Note that with the short sales constraint, the Harrison-Krpes model cannot generate panics.

To highlight the differences between our model and the Harrison-Kreps model, we revisit the example of a panic presented in subsection 2.1, by imposing the short sales constraint. In that example, we have demonstrated that a panic can occur without the short sales constraint or that the stock price today can be below all of the investors' valuations. We next show that in the Harrison-Kreps framework with risk neutrality and the short sales constraint, a bubble rather than a panic will occur.

We first determine the stock prices at time 1. When the realization is A, the price is equal to the maximum expectation among the investors, which is given by 6.4, the type j

investors' expectation. When the realization is B, the price is equal to 6, which is given by type i's expectation. The stock price at time 0 is then given by 6.2, which is a bubble price.

3.1 Panics and the Short Sales Constraint

With the short sales constraint, stock panics do not occur in both Miller (1977) and Harrison and Kreps (1978). In Miller, the short sales constraint prevents pessimistic investors from participating in the stock trading, and as a result, the equilibrium price reflects the views of the more optimistic investors. In Harrison and Kreps, the price is determined by the most optimistic investor in every state. In both models investors do not have an option to buy back the stock later at a low price.

We here demonstrate that even with the short sales constraint, a panic can still occur with risk averse investors. This result suggests that the short sales constraint is neither necessary nor sufficient for bubbles or panics to occur. For tractability, we use a two-investor, two-state economy.

Assume that for investor i the probability of realization A is denoted by $q_{iA} \equiv q > 1/2$, and the probability of realization B is denoted by $q_{iB} = (1-q) < 1/2$. Conditional on state A, the probability of the high value state (H) is $q_{iAH} = q$ and conditional on state B, the probability of the high value state is $q_{iBH} = 1-q$. Similarly, we have $q_{jA} = 1-q$, $q_{jB} = q$, $q_{jAH} = 1-q$, and $q_{jBH} = q$. Further assume that all investors possess the mean variance utility function and that the payoffs in states H and L (the low value state) are given by $V_H = 1$ and $V_L = 0$, respectively.

We solve the equilibrium using backward induction. We first solve for the stock price for the second period. At time 1, investor i's maximization problem is given by

$$\max_{D_{1i}} \left\{ D_{1i}[\mathrm{E}_{1i}[v] - P_1] - \frac{\gamma}{2}\mathrm{Var}_{1i}[v] D_{1i}^2 \right\}, \tag{21}$$

subject to the short sales constraint that $D_{1i} > 0$, where D_{1i} denotes investor i's demand for the stock and $E_{1i}[v] - P_1$ represents the expected profit of owning the stock. The optimal demand is then given by

$$D_{1i} = \frac{(\mathrm{E}_{1i}[v] - P_1)^+}{\gamma \mathrm{Var}_{1i}[v]},$$

where " + " denotes the positive part of a real number. Notice that under our simplified assumptions, the conditional expectations and variances are given by

$$\mathrm{E}_{1i}[v|A] = q, \quad \mathrm{E}_{1j}[v|B] = 1 - q,$$

$$\mathrm{Var}_{1i}[v|A] = \mathrm{Var}_{1j}[v|A] = \mathrm{Var}_{1i}[v|B] = \mathrm{Var}_{1j}[v|B] = q(1-q).$$

When state A is realized, investor i is more optimistic and the short sales constraint will not be binding for investor i but the constraint could be binding for investor j. When the short sales constraint does not bind for investor j, we have

$$D_{1iA} = \frac{q - P_{1A}}{\gamma q(1-q)}, \quad D_{1jA} = \frac{1 - q - P_{1A}}{\gamma q(1-q)}.$$

Applying the market clearing condition, $x = 1/2 D_{1iA} + 1/2 D_{1jA}$,[7] we obtain the equilibrium stock price:

$$P_{1A} = \frac{1}{2} - \gamma q(1-q)x. \tag{22}$$

From the expression for D_{1jA}, when the short sales constraint does not bind or $D_{1jA} > 0$, we must have

$$1 - q - P_{1A} \geq 0 \quad \text{or} \quad \gamma q(1-q)x > q - \frac{1}{2}.$$

On the other hand, when

$$\gamma q(1-q)x \leq q - \frac{1}{2},$$

the short sales constraint binds. We then have

$$D_{1iA} = \frac{q - P_{1A}}{\gamma q(1-q)}, \quad D_{1j} = \frac{(1 - q - P_{1A})^+}{\gamma q(1-q)} = 0.$$

[7] Note that x is the per capital supply of the stock. 1/2 is from the assumption that both types of investors are equally populated.

The market clearing condition yields

$$P_{1A} = q - 2\gamma q(1-q)x. \tag{23}$$

We can express the equilibrium stock price in a compact form as follows:

$$P_{1A} = \frac{1}{2} - \gamma q(1-q)x + \left[q - \frac{1}{2} - \gamma q(1-q)x\right]^+. \tag{24}$$

Due to the assumption of symmetry between states A and B, the equilibrium stock price, when state B is realized, is equal to P_{1A}, that is,

$$P_{1B} = \frac{1}{2} - \gamma q(1-q)x + \left[q - \frac{1}{2} - \gamma q(1-q)x\right]^+.$$

Since the risk free rate is assumed to be zero, we have that the equilibrium stock price at time 0 is given by

$$P_0 = \frac{1}{2} - \gamma q(1-q)x + \left[q - \frac{1}{2} - \gamma q(1-q)x\right]^+. \tag{25}$$

Notice that in the homogeneous belief case in which all investors share the same belief as investor i, investor i holds the total supply of the stock, x, under all conditions. Following the same procedure as in the case of heterogeneous beliefs, we have

$$P_{1iA} = q - \gamma q(1-q)x, \quad P_{1iB} = 1 - q - \gamma q(1-q)x,$$

$$\begin{aligned} P_{0i} &= P_{1iA} + (1-q)P_{1iB} - \gamma q(1-q)(P_{1iA} - P_{1iB})^2 x \\ &= 1 - 2q + 2q^2 - \gamma q(1-q)x - \gamma a(1-q)(2q-1)^2 x = P_{0j}. \end{aligned}$$

For a panic to occur or for P_0 to be lower than P_{0i}, we must have

$$\frac{1}{2} - \gamma q(1-q)x + (q - \frac{1}{2} - \gamma q(1-q)x)^+ < 1 - 2q + 2q^2 - \gamma q(1-q)x - \gamma a(1-q)(2q-1)^2 x,$$

which reduces to

$$(2q-1)^2\left[\frac{1}{2}-\gamma xq(1-q)\right] > \left[q-\frac{1}{2}-\gamma q(1-q)x\right]^+ . \tag{26}$$

Based on inequality (26), we consider three cases that contrast our model with those of Miller and Harrison and Kreps.

(1) We first examine the case in which investors are close to risk neutral or the term $\gamma xq(1-q)$ is negligible. Inequality (26) reduces to

$$\frac{1}{2}(2q-1)^2 > q-\frac{1}{2} > 0 \quad \text{or} \quad \left(q-\frac{1}{2}\right) > \frac{1}{2},$$

which cannot be satisfied because of the assumption that $1/2 < q < 1$. In other words, panics will never occur in the Harrison-Kreps model.

(2) We consider the case in which $\left[q-\frac{1}{2}-\gamma q(1-q)x\right]^+ > 0$ or the short sales constraint binds for investor j. In this case, $rq(1-q)x < 1/2$. Rearranging expression (26), we have

$$[1-(2q-1)^2]\gamma xq(1-q) > \left(q-\frac{1}{2}\right)(2-2q),$$

which reduces to

$$\gamma xq(1-q) > \frac{q-\frac{1}{2}}{2q}.$$

Consequently, when

$$\frac{q-\frac{1}{2}}{2q^2(1-q)} < \gamma x < \frac{q-\frac{1}{2}}{q(1-q)},$$

panics occur even when the short sales constraint binds. Note that the bounds depend on our normalization of $V_H = 1$.

(3) When the short sales constraint does not bind, the equilibrium stock price is lower than the price when the constraint binds, so it is easier to obtain panics in this case than in case 2.

We summarize the results obtained above in the following theorem.

Theorem 1 *When investors are risk averse, panics can still occur in the presence of the short sales constraint.*

Recall that we have demonstrated that without the short sales constraint, both bubbles and panics can arise even when the total supply of the stock is zero or investors are close to risk neutral so that the risk premium term in the stock price is negligible. This theorem presents a somewhat striking result, that is, even if the short sales constraint binds in both states at time 1, panics can still occur. This result holds only when investors are risk averse or when the total supply of the stock is nonzero.

Suppose that the short sales constraint binds for investor i in state B and for investor j in state A. At time 1, if state A occurs, with the short sales constraint binding for investor j, then investor i must hold all of the stock supply for the market to clear. The stock price can be very low because investor i may demand a high risk premium for holding the entire supply of the stock.[8] If state B occurs at time 1, then the stock price is higher than investor i's conditional expectation. At time 0, however, the weighted average of the stock prices in states A and B, adjusted by investor i's conditional precisions, can be lower than the stock price at time 0 when both types of investors share the same beliefs as investor i. Similarly, at time 0, the weighted average of the stock prices in states A and B, adjusted by investor j's conditional precisions, can be lower than the stock price at time 0 when both investors share the same beliefs as investor j. In other words, panics can occur at time 0 even when the short sales constraint binds in both states at time 1.

In a one-period arrangement, if the short sales constraint binds for some investors, then the stock price must be higher than those investors' expected values of the stock. Panics will never occur in this case.

[8] When investors i and j have homogeneous beliefs, the short sales constraint is never binding in order for the market to clear. In other words, both investors share the stock-related risk, and as a result, due to a lower risk premium, the stock price can be higher than that when investors have heterogeneous beliefs.

4 A Dynamic Sequential Equilibrium

So far we have obtained the result that asset bubbles and panics can arise without the short sales constraint when investors are myopic. In this section, we show that the investors' myopic nature is not crucial for bubbles and panics to occur.

We consider a fully dynamic model in which investors are forward looking and take into account both immediate capital gains and future returns. To obtain closed form solutions, we assume that there is a continuum of risk averse investors with negative exponential utility functions, $-\exp(-\gamma W_{2i})$, where γ is the investors' risk aversion coefficient and W_{2i} is investor i's terminal wealth or consumption. We consider a three-period model. Let v denote the value of the asset at the end of period 2. Assume that v is normally distributed and that for investor i, the unconditional mean of v is μ_i and the unconditional precision of v is h_i.[9] Investors trade in periods 0 and 1. At time 1, a public signal y arrives. Investors have different interpretations about the relationship between v and y, which generates trades among investors. In particular, investor i believes the following relationship:

$$a_i y - m_i = v + \epsilon_i, \ \ i \in [0, 1], \tag{27}$$

where a_i and m_i are constants and ϵ_i is normally distributed with mean 0 and precision n_i. a_i captures investor i's belief on how sensitive v changes with y, m_i/a_i is the constant term in the regression of y on v, and n_i represents investor i's confidence in the informativeness of y. With some normalization, we assume that $\int_i n_i m_i di = 0$, and $\int_i a_i n_i di = n$.[10] We further define $h \equiv \int_i h_i di$, $\int_i n_i di = n$, and $\mu \equiv \int_i h_i \mu_i di/h$.

We solve this dynamic maximization problem backward. We first solve the problem at time 1 for the second period. We then take the solutions for the second period as given and use them to solve the problem for the first period. We next provide a few key steps that are

[9] Similar results can also be obtained in closed form when the asset payoff and the signal have binomial distributions. However, the normality assumption allows the stock price to be expressed in terms of means and variances which make it easier to understand the intuition behind the results. We focus on the normality case here but the results for the binomial case are available on request.

[10] We can always redefine the signals for these normalization conditions to be satisfied. Suppose that $\int_i m_i n_i di = \bar{m}n$ and $\int_i a_i n_i di = \bar{a}n$. Let $y' = \bar{a}y - \bar{m}$, $a'_i = a_i/\bar{a}$, and $m'_i = m_i - a_i\bar{m}/\bar{a}$. Then $a'_i y' - m'_i = a_i y - m_i = v + \epsilon_i$ and we have $\int_i a'_i n_i di = n$ and $\int_i n_i m'_i di = 0$.

necessary to understand the equilibrium results to be presented in Theorem 2.

At time 1, there is only one period left, we obtain the following equilibrium price:

$$P_1 = \frac{h\mu + ny}{h+n} - \frac{\gamma x}{(h+n)},$$

that is, the price is the precision weighted average expectation minus the risk premium. At time 0, the expected utility for investor i has the following form:

$$\begin{aligned}
\mathrm{E}_{1i}[U_i] &= -\exp\left\{-\gamma\left[W_{0i} + D_{0i}(P_1 - P_0) + D_{1i}(\mathrm{E}_{1i}[v] - P_1)\right] + \frac{\gamma^2}{2}\mathrm{Var}_{1i}[v]D_{1i}^2\right\} \\
&= -\exp\left\{-\gamma\left[W_{0i} + D_{0i}(P_1 - P_0)\right] - \frac{h_i + n_i}{2}\left[\frac{h_i\mu_i + n_i(a_i y - m_i)}{h_i + n_i}\right.\right. \\
&\quad \left.\left. - \frac{h\mu + ny}{h+n} + \frac{\gamma x}{(h+n)}\right]^2\right\}. \qquad (28)
\end{aligned}$$

Taking the expectation with respect to y at time 0, we have

$$\mathrm{E}_{0i}[U_i] = \mathrm{E}_{0i}[\mathrm{E}_{1i}[U_i]] \propto -\int_y \exp\left[-\frac{(a_i y - \mu_i - m_i)^2}{2(h_i + n_i)/h_i n_i}\right]\mathrm{E}_{1i}[U_i]dy. \qquad (29)$$

Combining terms, we can rewrite the expected utility as

$$\begin{aligned}
\mathrm{E}_{0i}[U_i] &\propto -\int_y \exp[-\alpha_i y^2 - \beta_i y - \delta_i]dy \\
&= -\int_y \exp[-\alpha_i\left(y - \frac{\beta_i}{2\alpha_i}\right)^2 + \frac{\beta_i^2}{4\alpha_i} - \delta_i]dy \\
&\propto -\exp\left[\frac{\beta_i^2}{4\alpha_i} - \delta_i\right], \qquad (30)
\end{aligned}$$

where the expressions for α_i, β_i, and δ_i are given by

$$\alpha_i = \left[\frac{a_i^2 n_i}{2} - \frac{a_i n_i n}{h+n} + \frac{n^2(h_i + n_i)}{2(h+n)^2}\right], \qquad (31)$$

$$\beta_i = \left(D_{0i}\frac{\gamma n}{h+n} - \frac{a_i h_i n_i (m_i+\mu_i)}{h_i+n_i} \right.$$

$$\left. +(h_i+n_i)\left[\frac{h_i\mu_i - n_i m_i}{h_i+n_i} - \frac{h\mu - \gamma x}{h+n}\right]\left[\frac{a_i n_i}{h_i+n_i} - \frac{n}{h+n}\right]\right), \tag{32}$$

$$\delta_i = \gamma D_{0i}\left[\frac{h\mu-\gamma x}{h+n} - P_0\right]. \tag{33}$$

Dropping irrelevant terms, investor i's expected utility is proportional to

$$\mathrm{E}_{0i}[U_i] \propto -\exp\left[\frac{\beta_i^2 - 4\alpha_i\delta_i}{4\alpha_i}\right]. \tag{34}$$

The objective of investor i is to maximize $\mathrm{E}_{0i}[U_i]$ with respect to his demand function D_{0i}. The first-order condition is given by

$$\frac{\beta_i}{2\alpha_i}\frac{\partial\beta_i}{\partial D_{0i}} - \frac{\partial\delta_i}{\partial D_{0i}} = 0, \tag{35}$$

where

$$\frac{\partial\beta_i}{\partial D_{0i}} = \frac{\gamma n}{h+n}, \quad \frac{\partial\delta_i}{\partial D_{0i}} = \gamma\left[\frac{h\mu-\gamma x}{h+n} - P_0\right]. \tag{36}$$

The optimal demand can then be determined. The following theorem summarizes the equilibrium results.

Theorem 2 *There exists a dynamic sequential equilibrium in which the stock price and demand in the first period are given by*

$$D_{0i} = \frac{(h+n)^2}{\gamma n^2}\left\{\left[\frac{h\mu-\gamma x}{h+n} - P_0\right]\left[a_i^2 n_i - \frac{2a_i n_i n}{h+n} + \frac{n^2(h_i+n_i)}{(h+n)^2}\right]\right.$$

$$+\frac{n}{h+n}\left[a_i n_i m_i + a_i n_i\frac{h\mu}{h+n} + \frac{n}{h+n}\left[h_i\mu_i - n_i m_i - \frac{h\mu(h_i+n_i)}{h+n}\right]\right.$$

$$\left.\left. +\frac{\gamma x(h_i+n_i)}{(h+n)}\left[\frac{a_i n_i}{h_i+n_i} - \frac{n}{h+n}\right]\right]\right\}, \tag{37}$$

$$P_0 = \frac{(h\mu - \gamma x)\int_i a_i^2 n_i di + n\int_i a_i n_i m_i di}{h\int_i a_i^2 n_i di + n\int_i (a_i - 1)^2 n_i di}. \tag{38}$$

The equilibrium stock price and demand in the second period are given by

$$P_1 = \frac{h\mu + ny}{h+n} - \frac{\gamma x}{(h+n)}, \tag{39}$$

$$D_{1i} = \frac{1}{\gamma}[h_i\mu_i + n_i(a_i y - m_i) - (h_i + n_i)P_1]. \tag{40}$$

The detailed proof of this theorem is given in the appendix. It can be shown that in the homogeneous belief case in which all investors share the same belief as investor i, the equilibrium stock price would be determined according to investor i's belief about the asset value. It can be shown that the equilibrium stock price in the homogeneous case is given by

$$P_{0i} = \mu_i - \gamma x / h_i,$$

where $\mu_i = \mathrm{E}_{0i}[v]$. Let

$$P_{Max} = \max_i\{P_{0i}\}, \quad P_{Min} = \min_i\{P_{0i}\}.$$

For a bubble to occur, we must have

$$P_0 > P_{Max},$$

which reduces to

$$n\int_i a_i n_i m_i di > \left[h\int_i a_i^2 n_i di + n\int_i (a_i - 1)^2 n_i di\right] P_{Max} - (h\mu - \gamma x)\int_i a_i^2 n_i di. \tag{41}$$

For a panic to occur, we must have

$$P_0 < P_{Min},$$

which reduces to

$$n\int_i a_i n_i m_i di < \left[h\int_i a_i^2 n_i di + n\int_i (a_i - 1)^2 n_i di\right] P_{Min} - (h\mu - \gamma x)\int_i a_i^2 n_i di. \tag{42}$$

Thus, the condition for bubbles or panics to occur boils down to the magnitude of $\int_i a_i m_i di$. When $\int_i a_i m_i di$ is very large, a bubble will occur. When it is very small, a panic will result. Moreover, notice that the left hand side of (41) or (42) depends on m_i, whereas the right hand side does not depend on m_i. Consequently, one can always adjust m_i so that the inequality is satisfied. For example, suppose that there are two groups of investors with equal proportion. The first group believes that $1.6y = 5 + v + \epsilon_1$ where the variance of ϵ_1 is 1. The second group believes that $0.4y = v - 3 - \epsilon_2$ where the variance of ϵ_2 is also 1. Assume that $\mu_1 = 0, \mu_2 = 3, n_1 = n_2 = h = 1, \gamma = 1, x = 1$. A simple calculation shows that inequality (41) holds. The first group of investors believes that signal y has a high mean of 3.125 and a low variance of 0.39 while the second group believes that the signal has a low mean of 0 and a high variance of 6.25. Initially, the second group of investors is more optimistic than the first group. However, the first group is very confident that y will be around 3.125 and that with this realization of y, the second group's conditional expectation of the asset payoffs, $E_{12}[v|y = 3.125]$, is given by $(3 + 3.125 \times 0.4 + 3)/2 = 3.625$, where we have used the assumptions that v and y are normally distributed and that the precisions of v and ϵ are 1. It means that even though the first group is very pessimistic about the final payoffs, it believes that the second group is likely to become very optimistic. As a result, the first group of investors buys very aggressively in the first period so that the stock price can be above the intrinsic value of the second group of investors, which is 3. In other words, the first group buys the stock at a high price today expecting to sell the stock at an even higher price in the next period.

We next demonstrate that if a group of investors believes that the stock price will be high in the next period, then this group is willing to pay a high price at time 0. To simplify the discussion, we further assume that $n_i = n$, $h_i = h$, and $x = 0$, where $x = 0$ removes the risk premium term from the stock price function. Under these conditions, the equilibrium stock price at time 0 reduces to

$$P_0 = \frac{h\mu \int_i a_i^2 di + n \int_i a_i m_i di}{h \int_i a_i^2 di + n \int_i (a_i - 1)^2 di}. \tag{43}$$

In the homogeneous case in which all investors share the same belief as investor i, the equilibrium stock price is given by

$$P_{0i} = \mu_i.$$

Let $\mu_{Max} = \max_i\{\mu_i\}$ and $\mu_{Min} = \min_i\{\mu_i\}$, then $P_{Max} = \mu_{Max}$ and $P_{Min} = \mu_{Min}$.

The condition for the existence of a bubble becomes

$$P_0 > P_{Max} = \mu_{Max},$$

which reduces to

$$n \int_i a_i m_i di > \left[h \int_i a_i^2 di + n \int_i (a_i - 1)^2 di \right] \mu_{Max} - h\mu \int_i a_i^2 di.$$

Similarly, for a panic to occur, we must have

$$P_0 < P_{Min} = \mu_{Min},$$

which reduces to

$$n \int_i a_i m_i di < \left[h \int_i a_i^2 di + n \int_i (a_i - 1)^2 di \right] \mu_{Min} - h\mu \int_i a_i^2 di.$$

As shown in the myopic model, when conditional volatility is deterministic as in the normal distribution case, the law of iterated expectations must fail to get bubbles or panics. Intuitively, for a bubble to occur today, investors who buy the stock must believe that he can sell the stock at an even higher price in the future. To capture this notion, we examine the conditional expectation at time 0 of the stock prices at time 1, based on investor i's belief. We obtain that

$$\mathrm{E}_{0i}[P_1] = \frac{h\mu + n\mathrm{E}_{0i}[y]}{h+n} = \frac{h\mu + n(\mu_i + m_i)/a_i}{h+n}.$$

Notice that this conditional expectation is different from the unconditional expectation of investor i, which is given by μ_i. Indeed, it is possible that

$$\mathrm{E}_{0i}[P_1] = \frac{h\mu + nE_{0i}[y]}{h+n} = \frac{h\mu + n(\mu_i + m_i)/a_i}{h+n} > \mu_{Max},$$

which holds when

$$a_i m_i > \frac{a_i^2}{n}\left[(h+n)\mu_{Max} - h\mu\right] - a_i\mu_i. \tag{44}$$

Recall that in expression (44), $\mu_i = \mathrm{E}_{0i}[v]$, $\mu_{Max} = \max_i\{\mu_i\}$, and $\mu = \int_i h_i\mu_i di/h = \int_i \mu_i di$. If inequality (44) holds, then it means that some investors' time 0 expected value of the time 1 stock price is larger than the most optimistic investor's unconditional expected value of the stock payoffs. In other words, those investors expect the stock price to be higher in the next period so that they are willing to pay a higher price (than μ_{max}) today for the stock.

For an example, suppose that there are two types of investors, i and j, with equal proportion. Also suppose that $h = n = 1$ and that $\mu_i = 5$ and $\mu_j = 6$. We then have $\mu = 1/2 * (5+6) = 5.5$ and $\mu_{Max} = 6$. Inequality (44) becomes

$$a_i m_i > 5.5a_i^2 - 5a_i, \quad a_j m_j > 5.5a_j^2 - 6a_j.$$

It can be seen that there are numerous sets of a's and m's that satisfy the two inequalities. Under those conditions, both investors expect the next period price to be higher than 6 so that the stock price today is higher than 6.

Notice that the precision of P_1 for investor i is given by $a_i^2 n/[h(h+n)]$. In the special case in which $\mu_i = \mu$ for all i, the precision weighted average of P_1 across all investors at time 0 is given by

$$\mu[P_1] \equiv \frac{\int_i [a_i^2 h\mu + na_i\mu + na_i m_i]di}{\int_i a_i^2(h+n)di}. \tag{45}$$

When the bubble condition (41) holds, the difference between $\mu[P_1]$ and μ is given by

$$\begin{aligned}\mu[P_1] - \mu &= \frac{\int_i [a_i^2 h\mu + na_i\mu + na_i m_i]di}{\int_i a_i^2(h+n)di} - \mu \\ &= \frac{\int_i [na_i m_i - n(a_i-1)^2\mu]di}{\int_i a_i^2(h+n)di} > 0. \end{aligned} \tag{46}$$

Intuitively, a bubble occurs when the precision weighted average of P_1 is higher than the unconditional expectation of all investors. Notice that P_1 is also the precision weighted average of expectations of all investors for v at time 1. Thus, due to the violation of the law of iterated expectations across investors, a bubble occurs when some investors believe that the investors on average are very optimistic in the next period.

Similarly, it is possible that

$$\mathrm{E}_{0i}[P_1] = \frac{h\mu + n\mathrm{E}_{0i}[y]}{h+n} = \frac{h\mu + n(\mu_i + m_i)/a_i}{h+n} < \mu_{Min},$$

which holds when

$$a_i m_i < \frac{a_i^2}{n}[(h+n)\mu_{Min} - h\mu] - a_i\mu_i.$$

There are some papers in the literature that employ risk-averse investors and without the short sales constraint. See, for example, Kandel and Pearson (1995), Allen, Morris, and Shin (2004), and Cao and Ou-Yang (2004). Those papers, however, do not arrive at bubbles and panics. We next offer a necessary condition for the existence of bubbles and panics. It shall be seen that none of the papers on both difference of opinion and asymmetric information satisfy this condition.

Theorem 3 *If investors interpret the signals according to Equation (27), then a necessary condition for bubbles and panics to occur in a normal-exponential framework is that some of the a_i's in Equation (27) are different from 1.*

Proof: Start with the expression for P_0:

$$P_0 = \frac{h\mu \int_i a_i^2 di + n\int_i a_i m_i di}{h\int_i a_i^2 di + n\int_i (a_i - 1)^2 di}.$$

If $a_i = 1$, then we have that $\int_i a_i m_i di = \int_i m_i di = 0$ and that

$$P_0 = \mu, \quad \text{and} \quad P_{Min} \le P_0 \le P_{Max}.$$

Because μ is the precision weighted average of all investors' expectations, it cannot be higher (lower) than the highest (lowest) expectation. **Q.E.D.**

Previous models universally use the relation

$$y - m_i = v + \epsilon_i$$

in which investors differ about the value of m_i and the precision of ϵ_i. This is the reason that they are unable to arrive at bubbles and panics with risk-averse investors.

4.1 Bubbles and Panics with Almost Homogeneous Investors

We have shown that bubbles and panics can arise when investors are heterogeneous. A natural question is whether these results require investors to be very different from one another in their interpretations of information. We next show that bubbles and panics can occur even when investors' interpretations of information are only slightly different.

Suppose that ϵ denotes a very small number and that $l_i \equiv 1/h_i$ and $l \equiv 1/h$. Assume that $n_i = n$, $a_i = 1 + e_{ai}\sqrt{\epsilon}$, $m_i = e_{mi}\sqrt{\epsilon}$, $l_i = l + e_{li}\epsilon$, and $\mu_i = \bar{\mu} + e_{\mu i}\epsilon$. Let $\sigma_a^2 = \int_i e_{ai}^2 di$, $\sigma_m^2 = \int_i e_{mi} di$, $\rho_{am} = \int_i e_{ai}e_{mi} di/\sigma_a\sigma_m$, $\sigma_{Max} = \max_i(e_{\mu i} - \gamma x e_{li})$, and $\sigma_{Min} = \min_i(e_{\mu i} - \gamma x e_{li})$. Let $P_H \equiv \bar{\mu} - \gamma x/h = \bar{\mu} - \gamma l x$ denote the stock price that would obtain when all investors have homogeneous beliefs or when $\epsilon = 0$. We are interested in the conditions under which bubbles occurs when ϵ is very small or when investors are close to homogeneous.

For a given ϵ and the assumptions of investors' beliefs, condition (41) for a bubble to occur reduces to

$$n^2 \int_i e_{ai}e_{mi}\epsilon di > \left[h \int_i (1 + e_{ai}\sqrt{\epsilon})^2 n di + n^2 \int_i e_{ai}^2 \epsilon di\right] P_{Max} - nhP_H \int_i (1 + e_{ai}\sqrt{\epsilon})^2 di. \quad (47)$$

Dropping higher order terms of ϵ, the condition for a bubble to occur reduces to

$$n[\rho_{am}\sigma_a\sigma_m - \sigma_a^2 P_H]\epsilon > h\sigma_{Max}\epsilon. \quad (48)$$

Given ρ_{am}, σ_a, σ_{Max}, and P_H, the left hand side of (48) is linear in σ_m but the right hand side of (48) is independent of σ_m. Here the left hand side and the right hand side represent the investors' differences in their interpretations of the signal and their intrinsic value of the stock, respectively. Therefore, bubbles can result when σ_m is sufficiently large. Since ϵ can be arbitrarily small, bubbles can occur even when investors' beliefs are arbitrarily close to one another.

Similarly, for a very small ϵ, the condition for a panic to occur reduces to

$$n[\rho_{am}\sigma_a\sigma_m - \sigma_a^2 P_H]\epsilon < h\sigma_{Min}\epsilon. \tag{49}$$

We have shown that panics can exist when ϵ is in the neighborhood of zero such that investors are close to homogeneous. Nevertheless, for bubbles and panics to be sizable, ϵ has to be significantly different from zero and investors have to be significantly heterogeneous.

5 Conclusion

It has been widely believed or even taken for granted that the short sales constraint is crucial for asset bubbles to occur [Harrison and Kreps (1978)] and that the short sales constraint can cause the stock price to be biased upward [Miller (1977)]. In this paper, we demonstrate that the insight of Harrison and Kreps is still robust even without the short sales constraint. Investors' differences of opinion about the stock payoffs alone can lead them to believe that the stock price in the future will be even higher, so that they are willing to pay a higher (than the most optimistic investor's valuation) price today for the stock. Consequently, an asset bubble arises. Our model generalizes the Harrison-Kreps notion of a resale option with risk-averse investors but without the short sales constraint. We also demonstrate that asset panics can occur due to a buy-back option. In other words, differences of opinion can cause investors to believe that the future stock price will be even lower so that they are willing to sell the stock at a lower (than the most pessimistic investor's valuation) price today. We further show that the buy-back option may be so valuable that the current stock price can be lower than all investors' valuations even with the short sales constraint binding. This result suggests that Miller's intuition that the short sales constraint causes the stock price to be biased upward may not be robust in a dynamic setting with the possibility of a buy-back option.

Our results imply that it may be reasonable to observe that assets can be priced higher than the level that cannot be accounted for by shorting costs [Jones and Lamont (2002)] and that short interest may go down while certain asset bubbles are forming [Lamont and Stein (2004)]. Although shorting costs are generally very low already, they will inevitably be even lower in the future with the continual development of financial markets. Our exercise illustrates that both bubbles and panics may still arise if investors develop divergent views about the stock payoffs.

Appendix: Proof of Theorem 2

We solve this dynamic maximization problem backward. We first solve the problem at time 1 for the second period. We then take the solutions for the second period as given and use them to solve the problem for the first period.

At time 1, there is only one period left, investor i's conditional expectation of the stock payoff v is the precision weighted average of the unconditional mean and the signal:

$$\mathrm{E}_{1i}[v] = \frac{h_i\mu_i + n_i(v+\epsilon_i)}{h_i+n_i} = \frac{h_i\mu_i + n_i(a_i y - m_i)}{h_i+n_i},$$

and the conditional precision of v is the sum of the unconditional precision and the precision of the signal:

$$\mathrm{Var}_{1i}[v] = (h_i+n_i)^{-1}.$$

Let W_{1i} denote investor i's wealth at time 1. Investor i's optimization problem is given by

$$\max_{D_{1i}} \mathrm{E}_{1i}[U_i] = \max_{D_{1i}} \mathrm{E}_{1i}\left[-\exp(-\gamma W_{2i})\right] = \max_{D_{1i}} \mathrm{E}_{1i}\left[-\exp\left\{-\gamma\left[W_{1i} + D_{1i}(v-P_1)\right]\right\}\right],$$

where D_{1i} denotes investor i's demand for the risky stock, P_1 is the equilibrium stock price at time 1, and $W_{2i} = W_{1i} + D_{1i}(v-P_1)$ represents investor i's wealth or consumption at the terminal date $t=2$. Because v is normally distributed, this maximization problem reduces to a mean-variance problem:

$$\max_{D_{1i}} \left[-\exp\left\{-\gamma\left[W_{1i} + D_{1i}(\mathrm{E}_{1i}[v]-P_1)\right] + \frac{\gamma^2}{2}\mathrm{Var}_{1i}[v]D_{1i}^2\right\}\right].$$

The first-order condition (FOC) for the optimal demand at time 1 is given by

$$\mathrm{E}_{1i}[v] - P_1 - \gamma \mathrm{Var}_{1i}[v] D_{1i} = 0,$$

yielding

$$D_{1i} = \frac{\mathrm{E}_{1i}[v]-P_1}{\gamma \mathrm{Var}_{1i}[v]} = \frac{1}{\gamma}[h_i\mu_i + n_i(a_i y - m_i) - (h_i+n_i)P_1].$$

In equilibrium, the aggregate demand equals the supply. We have

$$x = \int_i D_{0i} di = \int_i \frac{1}{\gamma}[h_i \mu_i + n_i(a_i y - m_i) - (h_i + n_i)P_1] di = \frac{1}{\gamma}[h\mu - n - (h+n)P_1].$$

Notice that we have used the normalization conditions that $\int_i h_i di = h$, $\int_i n_i m_i di = 0$, $\int_i a_i n_i di = n$, $\int_i n_i di = n$, and $\int_i h_i \mu_i di = h\mu$. Solving for the equilibrium price, we arrive at

$$P_1 = \frac{h\mu + ny}{h+n} - \frac{\gamma x}{(h+n)},$$

that is, the price is the precision weighted average expectation minus a risk premium term determined by the investor's risk aversion and the supply of the risky stock.

We can now solve investor i's demand function D_{0i} at time 0 using backward induction. Investor i's wealth at time 1 is given by

$$W_{1i} = W_{0i} + D_{0i}(P_1 - P_0),$$

where P_0 is the equilibrium stock price at time 0. At time 1 investor i's expected utility over his terminal wealth has the following form:

$$\begin{aligned} \mathrm{E}_{1i}[U_i] &= -\exp\left\{-\gamma\left[W_{0i} + D_{0i}(P_1 - P_0) + D_{1i}(\mathrm{E}_{1i}[v] - P_1)\right] + \frac{\gamma^2}{2}\mathrm{Var}_{1i}[v]D_{1i}^2\right\} \\ &= -\exp\left\{-\gamma\left[W_{0i} + D_{0i}(P_1 - P_0)\right] - \frac{h_i + n_i}{2}\left[\frac{h_i\mu_i + n_i(a_i y - m_i)}{h_i + n_i}\right.\right. \\ &\quad \left.\left. -\frac{h\mu + ny}{h+n} + \frac{x}{\tau(h+n)}\right]^2\right\}. \end{aligned} \tag{50}$$

Taking the expectation with respect to y at time 0 and using the law of iterated expectations, we obtain investor i's expectation at time 0:

$$\mathrm{E}_{0i}[U_i] = \mathrm{E}_{0i}[\mathrm{E}_{1i}[U_i]] \propto -\int_y \exp\left[-\frac{(a_i y - \mu_i - m_i)^2}{2(h_i + n_i)/h_i n_i}\right] \mathrm{E}_{1i}[U_i] dy, \tag{51}$$

where the exponential function in the integral is the density function of y which is normally distributed. Combining terms, we can rewrite investor i's expected utility at time 0 as

$$\mathrm{E}_{0i}[U_i] \propto -\int_y \exp[-\alpha_i y^2 - \beta_i y - \delta_i] dy$$

$$= -\int_y \exp\left[-\alpha_i \left(y - \frac{\beta_i}{2\alpha_i}\right)^2 + \frac{\beta_i^2}{4\alpha_i} - \delta_i\right] dy \propto -\exp\left[\frac{\beta_i^2}{4\alpha_i} - \delta_i\right], \tag{52}$$

where the expressions for α_i, β_i, and δ_i are given by

$$\alpha_i = \left[\frac{a_i^2 n_i}{2} - \frac{a_i n_i n}{h+n} + \frac{n^2(h_i+n_i)}{2(h+n)^2}\right], \tag{53}$$

$$\beta_i = \left(\gamma D_{0i}\frac{n}{h+n} - \frac{a_i h_i n_i(m_i+\mu_i)}{h_i+n_i} + (h_i+n_i)\left[\frac{h_i\mu_i - n_i m_i}{h_i+n_i} - \frac{h\mu-\gamma x}{h+n}\right]\left[\frac{a_i n_i}{h_i+n_i} - \frac{n}{h+n}\right]\right), \tag{54}$$

$$\delta_i = \gamma D_{0i}\left[\frac{h\mu-\gamma x}{h+n} - P_0\right]. \tag{55}$$

Dropping irrelevant terms, investor i's expected utility is proportional to

$$E_{0i}[U_i] \propto -\exp\left[\frac{\beta_i^2 - 4\alpha_i\delta_i}{4\alpha_i}\right]. \tag{56}$$

The objective of investor i is to maximize $E_{0i}[U_i]$ with respect to his demand function D_{0i}. From Equation (56), the FOC is given by

$$\frac{\beta_i}{2\alpha_i}\frac{\partial\beta_i}{\partial D_{0i}} - \frac{\partial\delta_i}{\partial D_{0i}} = 0, \tag{57}$$

where

$$\frac{\partial\beta_i}{\partial D_{0i}} = \left(\gamma\frac{n}{h+n}\right), \quad \frac{\partial\delta_i}{\partial D_{0i}} = \gamma\left[\frac{h\mu-\gamma x}{h+n} - P_0\right]. \tag{58}$$

The FOC reduces to

$$
\begin{aligned}
&\frac{n}{h+n}\left[\gamma D_{0i}\frac{n}{h+n} - a_i n_i m_i - a_i n_i \frac{h\mu}{h+n} - \frac{n}{h+n}\left[h_i\mu_i - n_i m_i - \frac{h\mu(h_i+n_i)}{h+n}\right]\right. \\
&\left.+\frac{\gamma x(h_i+n_i)}{(h+n)}\left[\frac{a_i n_i}{h_i+n_i} - \frac{n}{h+n}\right]\right] \\
&-\left[\frac{h\mu-\gamma x}{h+n} - P_0\right]\left[a_i^2 n_i - \frac{2a_i n_i n}{h+n} + \frac{n^2(h_i+n_i)}{(h+n)^2}\right] = 0. \qquad (59)
\end{aligned}
$$

We then arrive at the optimal demand for investor i at time 0:

$$
\begin{aligned}
D_{0i} &= \frac{(h+n)^2}{\gamma n^2}\left\{\left[\frac{h\mu-\gamma x}{h+n} - P_0\right]\left[a_i^2 n_i - \frac{2a_i n_i n}{h+n} + \frac{n^2(h_i+n_i)}{(h+n)^2}\right]\right. \\
&+\frac{n}{h+n}\left[a_i n_i m_i + a_i n_i \frac{h\mu}{h+n} + \frac{n}{h+n}\left[h_i\mu_i - n_i m_i - \frac{h\mu(h_i+n_i)}{h+n}\right]\right. \\
&\left.\left.+\frac{\gamma x(h_i+n_i)}{(h+n)}\left[\frac{a_i n_i}{h_i+n_i} - \frac{n}{h+n}\right]\right]\right\}. \qquad (60)
\end{aligned}
$$

Using the market clearing condition, $x = \int_i D_{0i} di$, we have

$$
\begin{aligned}
x &= \int_i D_{0i} di = \frac{(h+n)^2}{\gamma n^2}\left\{\left[\frac{h\mu-\gamma x}{h+n} - P_0\right]\left[\int_i a_i^2 n_i di - \frac{n^2}{h+n}\right]\right. \\
&\left.+\frac{n}{h+n}\left[\int_i a_i n_i m_i di + n\frac{h\mu}{h+n}\right]\right\}, \qquad (61)
\end{aligned}
$$

which reduces to

$$
P_0\left[\int_i a_i^2 n_i di - \frac{n^2}{h+n}\right] = \frac{h\mu-\gamma x}{h+n}\left[\int_i a_i^2 n_i di\right] + \frac{n}{h+n}\int_i a_i n_i m_i di.
$$

Consequently, we arrive at the equilibrium stock price at time 0:

$$
P_0 = \frac{(h\mu-\gamma x)\int_i a_i^2 n_i di + n\int_i a_i n_i m_i di}{h\int_i a_i^2 n_i di + n\int_i (a_i-1)^2 n_i di}. \qquad (62)
$$

Q.E.D.

References

Abreu, D., and M. K. Brunnermeier, 2003, Bubbles and crashes, *Econometrica*, 71, 173-204.

Allen, F., and G. Gorton, 1993, Churning bubbles, *Review of Economic Studies*, 60, 813-836.

Allen, F., S. Morris, and A. Poslewaite, 1993, Finite bubbles with short sale constraints and asymmetric information, *Journal of Economic Theory*, 61, 206-229.

Allen, F., S. Morris, and H. S. Shin, 2004, Beauty contests, bubbles, and iterated expectations in asset markets, working paper, SSRN.

Biais, B., and P. Bossaerts, 1998, Asset prices and trading volume in a beauty contest, *Review of Economic Studies*, 65, 307-340.

Brunnermeier, M. K., and S. Nagel, 2004, Hedge funds and the technology bubble, *Journal of Finance*, 59, 2013-2040.

Cao, H., and H. Ou-Yang, 2004, Differences of opinion of public information and speculative trading in stocks and options, working paper, Duke and UNC.

D'Avolio, G., 2002, The market for borrowing stock, *Journal of Financial Economics*, 66, 271-306.

De Long, J. B., A. Shleifer, L. H. Summers, and R. J. Walsmann, 1990, Positive feedback investment strategies and destabilizing rational speculation, *Journal of Finance*, 45, 379-395.

DeMarzo, P. M., R. Kaniel, and I. Kremer, 2004, Relative wealth concerns and financial bubbles, working paper, Duke and Stanford.

Dhar, R., and W. N. Goetzmann, 2005, Bubble investors: What were they thinking? working paper, Yale.

Duffie, D., N. Garleanu, and L. Pederson, 2002, Securities lending, shorting, and pricing, *Journal of Financial Economics*, 66, 307-339.

Figlewski, S., and G. P. Webb, Options, short sales, and market completeness, *Journal of Finance*, 48, 761-777.

Harris, M., and A. Raviv, 1993, Differences of opinion make a horse race, *Review of Financial Studies*, 6, 473-506.

Harrison, J. M., and D. M. Kreps, 1978, Speculative investor behavior in a stock market with heterogeneous expectations, *Quarterly Journal of Economics*, 93, 323-336.

Hong, H., and J. C. Stein, 2003, Differences of opinion, short-sales constraints, and market crashes, *Review of Financial Studies*, 16, 487-525.

Jones, C. M., and O. A. Lamont, 2002, Short-sale constraints and stock returns, *Journal of Financial Economics*, 66, 207-239.

Kandel, E., and N. D. Pearson, 1995, Differential interpretation of public signals and trade in speculative markets, *Journal of Political Economy*, 103, 831-872.

Kyle, A. S., and T. Lin, 2002, Continuous trading with heterogeneous beliefs and no noise trading, working paper, Duke University and University of Hong Kong.

Lamont, O. A., and J. C. Stein, 2004, Aggregate short interest and market valuations, *American Economic Review*, 94, 29-32.

Lamont, O. A., and R. H. Thaler, 2003, Can the market add and subtract? Mispricing in tech stock carve-outs, *Journal of Political Economy*, 111, 227-268.

Miller, E. M., 1977, Risk, uncertainty, and divergence of opinion, *Journal of Finance*, 32, 1151-1168.

Morris, S., 1996, Speculative investor behavior and learning, *Quarterly Journal of Economics*, 111, 1111-1133.

Ofek, E., and M. Richardson, 2003, Dotcom mania: The rise and fall of internet stock prices, *Journal of Finance*, 58, 1113-1137.

Reed, A., 2001, Costly short-selling and stock price adjustment to earnings announcements, working paper, UNC.

Rubinstein, M. E., Great moments in financial economics III: Short-sales and stock prices, *Journal of Investment Management*, 2, 2004.

Scheinkman, J., and W. Xiong, 2003, Overconfidence, short-sale constraints, and bubbles, *Journal of Political Economy*, 111, 1183-1219.

Scheinkman, J., and W. Xiong, 2004, Heterogenous beliefs, speculation and trading in financial market, working paper, Princeton.

Spiegel, M., 2004, 2000 a bubble? 2002 a panic? Maybe nothing, *Wilmott Magazine*, March, 12-13.

Vissing-Jorgensen, A., Perspectives on behavioral finance: Does "irrationality" disappear with wealth? Evidence from expectations and actions, NBER Macroeconomics Annual, 2003.

Viswanathan, S., 2002, Strategic trading, heterogeneous beliefs, and short constraints, working paper, Duke University.