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Time-Varying Risk, Interest Rates, and Exchange Rates in General Equilibrium*

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ABSTRACT

Time-varying risk is the primary force driving nominal interest rate differentials on currency-denominated bonds. This finding is an immediate implication of the fact that exchange rates are roughly random walks. We show that a general equilibrium model with an endogenous source of risk variation—a variable degree of asset market segmentation—can produce many of the features of interest rates and exchange rates. The endogenous segmentation arises from a fixed cost for agents to exchange money for assets. As inflation varies, the benefit of asset market participation varies, and that changes the fraction of agents participating. These effects lead the risk premium to vary systematically with the level of inflation. An attractive feature of our model is that it produces variation in the risk premium even though the primitive shocks have constant conditional variances.

*The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

> Overall, the new view of finance amounts to a profound change. We have to get used to the fact that most returns and price variation comes from variation in *risk premia*.

Cochrane's (2001, p. 451) observation directs our attention to a critical counterfactual part of the standard monetary general equilibrium model: constant risk premia. Time-varying risk is essential for understanding movements in asset prices; that has been widely documented. We develop a model that can generate time-varying risk premia. The source of the model's variation in risk premia is its endogenous market segmentation; at any time, only a fraction of the model's agents choose to participate in the asset market. To illustrate the model's basic workings, we apply it to interest rates and exchange rates because data on those variables provide some of the most compelling evidence that variation in risk premia is a prime mover behind variation in asset prices.

To see why time-varying risk premia are important, consider the evidence that comes from nominal interest rates and exchange rates. As a concrete example, consider the risks faced by a U.S. investor choosing between bonds denominated in dollars or euros. Clearly, the dollar return on the euro bond is risky because next period's exchange rate is not known today. The *risk premium* compensates the holder of this bond for this exchange rate risk. In logs, this risk premium p_t is defined as the expected log dollar return on a euro bond minus the log dollar return on a dollar bond,

$$p_t = i_t^* + E_t \log e_{t+1} - \log e_t - i_t,$$

where i_t^* and i_t are the logarithms of euro and dollar gross interest rates, and e_t is the exchange rate. The difference in nominal interest rates across currencies can thus be divided into the expected change in the exchange rate between these currencies and a currency risk premium.

In standard equilibrium models of interest rates and exchange rates, risk premia are constant; interest rate differentials move one-for-one with the expected change in the exchange rate. However, nearly the opposite seems to happen in the data. A stylized view of the data is that exchange rates are roughly a random walk, in that the expected depreciation of a currency, $E_t \log e_{t+1} - \log e_t$, is roughly constant. (See, for example, Mussa 1986.) Thus, the interest rate differential, $i_t^* - i_t$, is approximately the risk premium p_t plus a constant.

Hence, the observed variations in the interest differential are almost entirely accounted for by movements in the risk premium.

A more nuanced view of the data is that when a currency's interest rate is high, that currency is expected to appreciate. This observation, documented by Fama (1984), Hodrick (1987), and Backus, Foresi, and Telmer (1995) among others, is widely referred to as the *forward premium anomaly.* This observation seems counterintuitive since one might expect investors to demand higher interest rates on currencies that are expected to fall in value, not those that are expected to rise. To explain the data, we need large fluctuations in risk premia—larger even than those in the interest differentials.

The basic idea that asset markets are segmented, in the sense that at any given time only a fraction of agents participate in them, has been used in a variety of settings to account for the high but constant levels of risk premia. (See, for example, Allen and Gale 1994, Basak and Cuoco 1998, and Alvarez and Jermann 2001.) Our monetary general equilibrium model goes beyond those attempts. The model generates variable risk premia as a result of variation in the degree of market segmentation that arises endogenously, in response to changes in the money growth rate. We show that the model can generate, qualitatively, the type of systematic variation in risk premia called for by the data on interest rates and exchange rates. Rather than build a quantitative model, we deliberately build a simple model in which the main mechanism can be clearly seen. For example, throughout, we abstract from trade in goods in order to focus on frictions in asset markets.

Our model is a two-country, pure exchange, cash-in-advance economy. The key difference between this model and the standard cash-in-advance model is that here agents must pay a fixed cost to transfer money between the goods market and the asset market. This fixed transfer cost is similar to that in the models of Baumol (1952) and Tobin (1956), and it differs across agents. In each period, agents with a fixed transfer cost below some cutoff level pay it and thus, at the margin, freely exchange money and bonds. Agents with a fixed transfer cost higher than the cutoff level choose not to pay it, so do not make these exchanges. In this sense, the asset market is segmented.

The mechanism through which this segmentation leads to variable risk premia is straightforward. Increases in the money growth rate increase the inflation rate, which increases the net benefit of participating in the asset market. An increase in money growth thus increases the fraction of agents that participate in the asset market, reduces segmentation, and hence lowers the risk premium. We show that this type of variable risk premium can be the primary force driving interest rate differentials and that it can generate the forward premium anomaly.

Our model also has implications beyond the time series data, for the cross-section data across countries. One such implication is that if inflation is permanently higher in one country, then asset market participation is too. With higher asset market participation, markets are less segmented; thus, the volatility of the risk premia should be small. The model thus predicts that countries with high enough inflation should not have a forward premium anomaly. This prediction is supported by Bansal and Dahlquist (2000), who study the forward premium in developed and emerging economies.

Our model is related to a huge literature on generating large and volatile risk premia in general equilibrium models. The work of Mehra and Prescott (1985) and Hansen and Jagannathan (1991) has established that in order to generate large risk premia, the general equilibrium model must produce extremely volatile pricing kernels. Also well-known is that because of the data's rather small variations in aggregate consumption, a representative agent model with standard utility functions cannot generate substantial risk premia. Therefore, attempts to account for foreign exchange risk premia in models of this type fail dramatically. (See Backus, Gregory, and Telmer 1993, Canova and Marrinan 1993, Bansal et al. 1995, Bekaert 1996, Engel 1996, and Obstfeld and Rogoff 2001.) Indeed, the only way such models could generate large and variable risk premia is by generating an implied series for aggregate consumption that is many times more variable and that has a variance that fluctuates much more than observed consumption.

Faced with these difficulties, the study of risk in general equilibrium models has split into two branches. One branch investigates new classes of utility functions that make the marginal utility of consumption extremely sensitive to small variations in consumption. The work of Campbell and Cochrane (1999) typifies this branch. Bekaert (1996) examines the ability of a model along these lines to generate large and variable foreign exchange risk premia. The other branch investigates limited participation models in which the consumption of the marginal investor is not equal to aggregate consumption. The work of Alvarez and Jermann (2001) and Lustig (2003) typifies this branch. Our work here is firmly part of this second branch. In our model, the consumption of the marginal investor is quite variable even though aggregate consumption is essentially constant.

A body of empirical work supports the idea that limited participation in asset markets is important in helping to account for empirical failures of consumption-based asset-pricing models. Mankiw and Zeldes (1991) argue that the consumption of asset market participants, defined as stockholders, is more volatile and more highly correlated with the excess return on the stockmarket than the consumption of nonparticipants. Brav, Constantinides, and Geczy (2002) argue that if attention is restricted to the consumption of active market participants, many standard asset-pricing puzzles, like the equity premium puzzle, can be partly accounted for in a consumption-based asset-pricing model with low and economically plausible values of the relative risk aversion coefficient. Vissing-Jorgensen (2002) provides similar evidence.

To keep the analysis simple in this paper, we take an extreme view of the limited participation idea. In our model aggregate consumption is (essentially) constant and hence plays no role in pricing risk. Instead this risk is priced by the marginal investor whose consumption is quite different from aggregate consumption. It is worth noting that Lustig and Verdelhan (2005) present some interesting evidence that aggregate U.S. consumption growth may be useful for pricing exchange rate risk. In a more complicated version of our model, we could have both aggregate consumption and the consumption of the marginal investor playing a role in pricing exchange rate risk.

As Backus, Foresi, and Telmer (1995) and Engel (1996) have emphasized, standard monetary models with standard utility functions have no chance of producing the forward premium anomaly because they generate a constant risk premium as long as the underlying driving processes have constant conditional variances. Backus, Foresi, and Telmer argue that empirically it is unlikely that this anomaly can be generated by primitive processes that have nonconstant conditional variances. (See also Hodrick 1989.) Instead, they argue, what is needed is a model that generates nonconstant risk premia from driving processes that have constant conditional variances. Our model does exactly that.

Our work builds on that of Rotemberg (1985) and Alvarez and Atkeson (1997) and is most closely related to that of Alvarez, Atkeson, and Kehoe (2002). It is also related to the work of Grilli and Roubini (1992) and Schlagenhauf and Wrase (1995), who study the effects of money injections on exchange rates in two-country variants of the models of Lucas (1990) and Fuerst (1992) but do not address variations in the risk premium.

1. Risk, Interest Rates, and Exchange Rates in the Data

Here we document that fluctuations in interest differentials are large, and we develop our argument that these fluctuations are driven mainly by time-varying risk.

Backus, Foresi, and Telmer (2001) compute statistics on the difference between monthly eurocurrency interest rates denominated in U.S. dollars and the corresponding interest rates for the other G-7 currencies over the time period July 1974 through November 1994. The average of the standard deviations of these interest differentials is large: 3.5 percentage points on an annualized basis. Moreover, these interest differentials are quite persistent: at a monthly level, the average of their first-order autocorrelations is .83.

To see that fluctuations in interest differentials are driven mainly by time-varying risk, define the (log) risk premium for a euro-denominated bond as the expected log dollar return on a euro bond minus the log dollar return on a dollar bond. Let i_t and i_t^* be the nominal interest rates on the dollar and euro bonds and e_t be the price of euros (foreign currency) in units of dollars (home currency), or the exchange rate between the currencies, in all period t. The dollar return on a euro bond, $(1+i_t^*)e_{t+1}/e_t$, is obtained by converting a dollar in period t to $1/e_t$ euros, buying a euro bond paying interest $1+i_t^*$, and then converting the resulting euros back to dollars in $t+1$ at the exchange rate e_{t+1}. Hence, in logs, the risk premium is

$$(1) \qquad p_t = i_t^* + E_t \log e_{t+1} - \log e_t - i_t.$$

Clearly, the dollar return on the euro bond is risky because the future exchange rate e_{t+1} is not known in t. The risk premium compensates the holder of the bond for this exchange rate risk.

To see our argument in its simplest form, suppose that the exchange rate is a random walk, so that $E_t \log e_{t+1} - \log e_t$ is constant. Then (1) implies that

$$(2) \qquad i_t - i_t^* = -p_t + E_t \log e_{t+1} - \log e_t$$

so that the interest differential is just the risk premium plus a constant. Hence, all of the movements in the interest differential are matched by corresponding movements in the risk premium, so that

$$\mathrm{var}(p_t) = \mathrm{var}(i_t - i_t^*).$$

In the data, however, exchange rates are only approximately random walks. In fact, one of the most puzzling features of exchange rate data is the tendency for high interest rate currencies to appreciate, in that

$$\text{(3)} \qquad \mathrm{cov}\left(i_t - i_t^*, \log e_{t+1} - \log e_t\right) \leq 0.$$

Notice that (3) is equivalent to

$$\text{(4)} \qquad \mathrm{cov}\left(i_t - i_t^*, E_t \log e_{t+1} - \log e_t\right) \leq 0.$$

Thus, (3) implies that exchange rates are not random walks because expected depreciation rates are correlated with interest differentials. This tendency for high interest rate currencies to appreciate has been widely documented for the currencies of the major industrialized countries over the period of floating exchange rates. (For a recent discussion, see, for example, Backus, Foresi, and Telmer 2001.) The inequality (3) is referred to as the forward premium anomaly.[1] In the literature, this anomaly is documented by a regression of the change in the exchange rates on the interest differential:

$$\text{(5)} \qquad \log e_{t+1} - \log e_t = a + b(i_t - i_t^*) + u_{t+1}.$$

Such regressions typically yield estimates of b that are zero or negative. We refer to b as the slope coefficient in the Fama regression.

This feature of the data is particularly puzzling because it implies that fluctuations in risk premia that are needed to account for fluctuations in interest differentials are even larger than those needed if exchange rates were random walks:

$$\text{(6)} \qquad \mathrm{var}\left(p_t\right) \geq \mathrm{var}\left(i_t - i_t^*\right).$$

To see that (4) implies (6), use (1) to rewrite (4) as $\mathrm{var}(i_t - i_t^*) + \mathrm{cov}(i_t - i_t^*, p_t) \leq 0$ or

$$\mathrm{var}\left(i_t - i_t^*\right) \leq -\mathrm{cov}\left(i_t - i_t^*, p_t\right) = -\mathrm{corr}\left(i_t - i_t^*, p_t\right) \mathrm{std}\left(i_t - i_t^*\right) \mathrm{std}\left(p_t\right).$$

Then divide by $\mathrm{std}(i_t - i_t^*)$, and use the fact that a correlation is less than or equal to one in absolute value.

2. The Economy

Now we describe our monetary general equilibrium model with segmented markets that generates time-varying risk premia.

A. An Outline

Consider a two-country, cash-in-advance economy with an infinite number of periods $t = 0, 1, 2, \ldots$ Call one country the *home* country and the other the *foreign* country. Each country has a government and a continuum of households of measure one. Households in the home country use the home currency, *dollars,* to purchase a home good. Households in the foreign country use the foreign currency, *euros,* to purchase a foreign good.

Trade in this economy in periods $t \geq 1$ occurs in three separate locations: an asset market and two goods markets, one in each country. In the asset market, households trade the two currencies and dollar and euro bonds, which promise delivery of the relevant currency in the asset market in the next period, and the two governments introduce their currencies via open market operations. In each goods market, households use the local currency to buy the local good subject to a cash-in-advance constraint and sell their endowment of the local good for local currency.

Each household must pay a real fixed cost γ for each transfer of cash between the asset market and the goods market. This fixed cost is constant over time for any specific household but varies across households in both countries according to a distribution with density $f(\gamma)$ and distribution $F(\gamma)$.[2] Households are indexed by their fixed cost γ. The fixed costs for households in each country are in units of the local good. We assume $F(0) > 0$, so that a positive mass of households has zero fixed costs.

The only source of uncertainty in this economy is the money growth shocks in the two countries. The timing within each period $t \geq 1$ for a household in the home country is illustrated in Figure 1. We emphasize the physical separation of the markets by separating them in the figure. Households in the home country enter the period with the cash $P_{-1}y$ they obtained from selling their home good endowments in $t-1$, where P_{-1} is the price level and y is their endowment. Each government conducts an open market operation in the asset market, which determines the realizations of money growth μ and μ^* in the two countries and the current price levels in the two countries P and P^*.

The household then splits into a worker and a shopper. Each period the worker sells the household endowment y for cash Py and rejoins the shopper at the end of the period. The shopper takes the household's cash $P_{-1}y$ with real value $n = P_{-1}y/P$ and shops for goods. The shopper can choose to pay the fixed cost γ to transfer an amount of cash Px with real value x to or from the asset market. This fixed cost is paid in cash obtained in the asset market. If the shopper pays the fixed cost, then the cash-in-advance constraint is $c = n + x$; otherwise, this constraint is $c = n$.

The household also enters the period with bonds that are claims to cash in the asset market with payoffs contingent on the rates of money growth μ and μ^* in the current period. This cash can be either reinvested in the asset market or, if the fixed cost is paid, transferred to the goods market. With B denoting the current realized value of the state-contingent bonds, q the price of bonds, and $\int qB'$ the household's purchases of new bonds, the asset market constraint is $B = \int qB' + P(x + \gamma)$ if the fixed cost is paid and $B = \int qB'$ otherwise. At the beginning of period $t+1$, this household starts with cash Py in the goods market and contingent bonds B' in the asset market.

In equilibrium, households with a sufficiently low fixed cost pay it and transfer cash between the goods and asset markets while others do not. We refer to households that pay the fixed cost as *active* and households that do not as *inactive*. Inactive households simply consume their current real balances.

B. Details

Now, we flesh out this outline of the economy.

Throughout the paper we assume that the shopper's cash-in-advance constraint binds and that in the asset market, households hold their assets in interest-bearing securities rather than cash. It is easy to provide sufficient conditions for these assumptions to hold. Essentially, if the average inflation rate is high enough, then money held over from one period to another in a goods market loses much of its value and households' cash-in-advance constraints bind. If nominal interest rates are positive, then bonds dominate cash held in the asset market and households hold their assets in interest-bearing securities rather than cash.

At the beginning of period 1, home households of type γ have M_0 units of home money (dollars), $\bar{B}_h(\gamma)$ units of the home government debt (bonds), and $\bar{B}_h^*$ units of the foreign government debt, which are claims on $\bar{B}_h(\gamma)$ dollars and $\bar{B}_h^*$ euros in the asset market in that period. Likewise, foreign households start period 1 with M_0^* euro holdings in the foreign goods market and start period 0 with $\bar{B}_f$ units of the home government debt and $\bar{B}_f^*(\gamma)$ units of the foreign government debt in the asset market.

Let M_t denote the stock of dollars in period t, and let $\mu_t = M_t/M_{t-1}$ denote the growth rate of this stock. Similarly, let μ_t^* be the growth rate of the stock of euros M_t^*. Let $s_t = (\mu_t, \mu_t^*)$ denote the aggregate event in period t. Then let $s^t = (s_1, \ldots, s_t)$ denote the history of money growth shocks through period t, and let $g(s^t)$ denote the density of the probability distribution over such histories.

The home government issues one-period dollar bonds contingent on the aggregate state s^t. In period t, given state s^t, the home government pays off outstanding bonds $B(s^t)$ in dollars and issues claims to dollars in the next asset market of the form $B(s^t, s_{t+1})$ at prices $q(s^t, s_{t+1})$. The home government budget constraint at s^t with $t \geq 1$ is

$$(7) \qquad B(s^t) = M(s^t) - M(s^{t-1}) + \int_{s_{t+1}} q(s^t, s_{t+1}) B(s^t, s_{t+1}) ds_{t+1}$$

with $M(s^0) = \bar{M}$ given, and in $t = 0$, the constraint is $\bar{B} = \int_{s_1} q(s^1) B(s^1)\, ds_1$. Likewise, the foreign government issues euro bonds denoted $B^*(s^t)$ with bond prices denoted $q^*(s^t, s_{t+1})$. The budget constraint for the foreign government is then analogous to the constraint above.

In the asset market in each period and state, home households trade a set of one-period dollar bonds and euro bonds that have payoffs next period contingent on the aggregate event s_{t+1}. Arbitrage between these bonds implies that

$$(8) \qquad q(s^t, s_{t+1}) = q^*(s^t, s_{t+1}) e(s^t)/e(s^{t+1}),$$

where $e(s^t)$ is the exchange rate for euros in terms of dollars in state s^t. Thus, without loss of generality, we can assume that home households trade in home bonds and foreign households trade in foreign bonds.

Consider now the problem of households of type γ in the home country. Let $P(s^t)$ denote the price level in dollars in the home goods market in period t. In each period $t \geq 1$, in the goods market, households of type γ start the period with dollar real balances $n(s^t,\gamma)$. They then choose transfers of real balances between the goods market and the asset market $x(s^t,\gamma)$, an indicator variable $z(s^t,\gamma)$ equal to zero if these transfers are zero and one if they are more than zero, and consumption of the home good $c(s^t,\gamma)$ subject to the cash-in-advance constraint and transition law

$$(9)\quad c(s^t,\gamma) = n(s^t,\gamma) + x(s^t,\gamma)z(s^t,\gamma)$$

$$(10)\quad n(s^{t+1},\gamma) = \frac{P(s^t)y}{P(s^{t+1})},$$

where in (9) in $t = 1$, the term $P(s^1)n(s^1,\gamma)$ is given by M_0. In the asset market in $t \geq 1$, home households begin with cash payments $B(s^t,\gamma)$ on their bonds. They purchase new bonds and make cash transfers to the goods market subject to the sequence of budget constraints

$$(11)\quad B(s^t,\gamma) = \int_{s_{t+1}} q(s^t,s_{t+1})B(s^t,s_{t+1})ds_{t+1} + P(s^t)\left[x(s^t,\gamma) + \gamma\right] z(s^t,\gamma).$$

Assume that both consumption $c(s^t,\gamma)$ and real bond holdings $B(s^t,\gamma)/P(s^t)$ are uniformly bounded by some large constants.

The problem of the home household of type γ is to maximize

$$(12)\quad \sum_{t=1}^{\infty}\beta^t \int U\big(c(s^t,\gamma)\big)g(s^t)\ d\mu^t$$

subject to the constraints (9)–(11). Households in the foreign country solve the analogous problem with $P^*(s^t)$ denoting the price level in the foreign country in euros. We require that $\int \bar{B}_h(\gamma)f(\gamma)\ d\gamma + \bar{B}_f = \bar{B}$ and $\bar{B}^*_h + \int \bar{B}^*_f(\gamma)f(\gamma)\ d\gamma = \bar{B}^*$.

Since each transfer of cash between the asset market and the home goods market consumes γ units of the home good, the total goods cost of carrying out all transfers between home households and the asset market in t is $\gamma \int z(s^t,\gamma)f(\gamma)\ d\gamma$, and likewise for the foreign households. The resource constraint in the home country is given by

$$(13)\quad \int \left[c(s^t,\gamma) + \gamma z(s^t,\gamma)\right] f(\gamma)\ d\gamma = Y$$

for all t,s^t, with the analogous constraint in the foreign country. The fixed costs are paid for with cash obtained in the asset market. Thus, the home country money market–clearing condition in $t \geq 0$ is given by

$$(14)\quad \int \Big(n(s^t,\gamma) + \left[x(s^t,\gamma) + \gamma\right] z(s^t,\gamma)\Big)\, f(\gamma)\ d\gamma = M(s^t)/P(s^t)$$

for all s^t. The money market–clearing conditions for the foreign country are analogous. We let c denote the sequences of functions $c(s^t,\gamma)$ and use similar notation for the other variables.

An *equilibrium* is a collection of bond and goods prices (q,q^*) and (P,P^*), together with bond holdings (B,B^*) and allocations for home and foreign households (c,x,z,n) and (c^*,x^*,z^*,n^*), such that for each γ, the bond holdings and the allocation solve the households' utility maximization problems, the governments' budget constraints hold, and the resource constraints and the money market–clearing conditions are satisfied.

3. Characterizing Equilibrium

Here we solve for the equilibrium consumption and real balances of active and inactive households, namely those that transfer cash between markets and those that do not. We then characterize the link between the consumption of active households and asset prices.

Under the assumption that the cash-in-advance constraint always binds, a household's decision to pay the fixed cost to trade in period t is static since this decision affects only the household's current consumption and bond holdings and not the real balances it holds later in the goods market. Notice that the constraints (10), (13), and (14) imply that the price level is

$$P(s^t) = M(s^t)/Y. \tag{15}$$

The inflation rate is $\pi_t = \mu_t$, and real money holdings are $n(s^t,\gamma) = y/\mu_t$. Hence, the consumption of inactive households is $c(s^t,\gamma) = y/\mu_t$. Let $c_A(s^t,\gamma)$ denote the consumption of an active household for a given s^t and γ.

In this economy, inflation is distorting because it reduces the consumption of any household that chooses to be inactive. This effect induces some households to use real resources to pay the fixed cost, thereby reducing the total amount of resources available for consumption. This is the only distortion in the model. Because of this feature, the competitive equilibrium allocations and asset prices can be found from the solution to the following planning problem for the home country, together with the analogous problem for the foreign country:

$$\max \sum_{t=1}^{\infty} \beta^t \int_{s^t} \int_{\gamma} U\left(c(s^t,\gamma)\right) f(\gamma) g(s^t)\ d\gamma ds^t$$

subject to the resource constraint (13) and

$$(16)\quad c(s^t,\gamma) = z(s^t,\gamma)c_A(s^t,\gamma) + [1 - z(s^t,\gamma)]y/\mu_t.$$

The constraint (16) captures the restriction that the consumption of households that do not pay the fixed cost is pinned down by their real money balances y/μ_t. Here the planning weight for households of type γ is simply the fraction of agents of this type.

This problem can be decentralized with the appropriate settings of the initial endowments $B(\gamma)$ and $B^*(\gamma)$. Asset prices are obtained from the multipliers on the resource constraints above.

Notice that this problem reduces to a sequence of static problems. We first analyze the consumption pattern for a fixed choice of z and then analyze the optimal choice of z.

The first-order condition for c_A reduces to

$$(17)\quad \beta^t U'\left(c_A(s^t,\gamma)\right) g(s^t) = \lambda(s^t),$$

where $\lambda(s^t)$ is the multiplier on the resource constraint. This first-order condition clearly implies that all households that pay the fixed cost choose the same consumption levels, which means that $c_A(s^t,\gamma)$ is independent of γ. Since this problem is static, this consumption level depends on only the current shock μ_t. Hence, we denote this consumption as $c_A(\mu_t)$.

Given that the solution to the planning problem depends on only current μ_t and γ, we drop dependence on t. It should be clear that the optimal choice of z has a cutoff rule form: for each shock μ, there is some fixed cost level $\bar{\gamma}(\mu)$ such that the households with $\gamma \leq \bar{\gamma}(\mu)$ pay this fixed cost and other households do not. For each μ, the planning problem thus reduces to choosing two numbers, $c_A(\mu)$ and $\bar{\gamma}(\mu)$, to solve

$$\max U\big(c_A(\mu)\big)F\big(\bar{\gamma}(\mu)\big) + U(y/\mu)\left[1 - F\big(\bar{\gamma}(\mu)\big)\right]$$

subject to

$$(18)\quad c_A(\mu)F\big(\bar{\gamma}(\mu)\big) + \int_0^{\bar{\gamma}(\mu)} \gamma f(\gamma)\, d\gamma + (y/\mu)\left[1 - F\big(\bar{\gamma}(\mu)\big)\right] = y.$$

The first-order conditions can be summarized by

$$(19)\quad U\big(c_A(\mu)\big) - U(y/\mu) - U'\big(c_A(\mu)\big)[c_A(\mu) + \bar{\gamma}(\mu) - (y/\mu)] = 0$$

and (18). In Appendix A, we show that the solution to these two equations, namely, $c_A(\mu)$ and $\bar{\gamma}(\mu)$, is unique. We then have the following proposition.

Proposition 1. The equilibrium consumption of households is given by

$$c(s^t,\gamma) = \begin{cases} y/\mu_t & \text{if} \quad \gamma \leq \bar{\gamma}(\mu_t) \\ c_A(\mu_t) & \text{otherwise,} \end{cases}$$

where the functions $\bar{\gamma}(\mu)$ and $c_A(\mu)$ are the solutions to (18) and (19).

In the decentralized economy corresponding to the planning problem, asset prices are given by the multipliers on the resource constraints for the planning problem. Here, from (17), these multipliers are equal to the marginal utility of active households.

Hence, the pricing kernel for dollar assets is

$$(20) \quad m(s^t,s_{t+1}) = \beta \frac{U'\big(c_A(\mu_{t+1})\big)}{U'\big(c_A(\mu_t)\big)} \frac{1}{\mu_{t+1}},$$

while the pricing kernel for euro assets is

$$(21) \quad m^*(s^t,s_{t+1}) = \beta \frac{U'\big(c_A^*(\mu_{t+1}^*)\big)}{U'\big(c_A^*(\mu_t^*)\big)} \frac{1}{\mu_{t+1}^*}.$$

These kernels can be thought of as the state-contingent prices for dollars and euros normalized by the probabilities of the state.

These pricing kernels can price any dollar or euro asset. In particular, the pricing kernels immediately imply that any asset purchased in period t with a dollar return of R_{t+1} between periods t and $t+1$ satisfies the Euler equation

$$(22) \quad 1 = E_t m_{t+1} R_{t+1},$$

where, for simplicity here and in much of what follows, we drop the s^t notation. Likewise, every possible euro asset with rate of return R_{t+1}^* from t to $t+1$ satisfies the Euler equation

$$(23) \quad 1 = E_t m_{t+1}^* R_{t+1}^*.$$

Noting that $\exp(i_t)$ is the dollar return on a dollar-denominated bond with interest rate i_t and $\exp(i_t^*)$ is the expected euro return on a euro-denominated bond with interest rate i_t^*, these Euler equations imply that

$$(24) \quad i_t = -\log E_t m_{t+1} \text{ and } i_t^* = -\log E_t m_{t+1}^*.$$

The pricing kernels for dollars and euros have a natural relation: $m^*_{t+1} = m_{t+1}e_{t+1}/e_t$. This can be seen as follows. Every euro asset with euro rate of return R^*_{t+1} has a corresponding dollar asset with rate of return $R_{t+1} = R^*_{t+1}e_{t+1}/e_t$ formed when a dollar investor converts dollars into euros in t, buys the euro asset, and converts the return back into dollars in $t+1$. Equilibrium requires that

$$(25) \quad 1 = E_t m_{t+1} R_{t+1} = E_t \left\{ \left[m_{t+1} \left(\frac{e_{t+1}}{e_t} \right) \right] R^*_{t+1} \right\}.$$

Since (25) holds for every euro return, $m_{t+1}e_{t+1}/e_t$ is an equilibrium pricing kernel for euro assets. Complete markets have only one euro pricing kernel, so

$$(26) \quad \log e_{t+1} - \log e_t = \log m^*_{t+1} - \log m_{t+1}.$$

Substituting (24) and (26) into our original expression for the risk premium, (1) gives that

$$(27) \quad p_t = E_t \log m^*_{t+1} - E_t \log m_{t+1} - (\log E_t m^*_{t+1} - \log E_t m_{t+1}).$$

Hence, the currency risk premium depends on the difference between the expected value of the log and the log of the expectation of the pricing kernel. Jensen's inequality implies that fluctuations in the risk premium are driven by fluctuations in the conditional variability of the pricing kernel.

Finally, note that given any period 0 exchange rate e_0, (26) together with the kernels gives the entire path of the nominal exchange rate e_t. It is easy to show that the period 0 nominal exchange rate e_0 is given by

$$(28) \quad e_0 = \left(\bar{B} - \bar{B}_h \right) / \bar{B}^*_h.$$

Clearly, this exchange rate exists and is positive as long as $\bar{B}_h < \bar{B}$ and $\bar{B}^*_h > 0$ or $\bar{B}_h > \bar{B}$ and $\bar{B}^*_h < 0$.

4. Active Household's Marginal Utility

In our model, the active households price assets in the sense that the pricing kernels (20) and (21) are determined by those households' marginal utilities. Thus, in order to characterize the link between money shocks and either exchange rates or interest rates, we need to determine how these marginal utilities respond to money shocks, or how $U'\big(c_A(\mu_t)\big)$ varies with μ_t.

In the simplest monetary models (such as in Lucas 1982), all the agents are active every period, and changes in money growth have no impact on marginal utilities. Our model introduces two key innovations to those simple models. First, here, because of the segmentation of asset markets, changes in money growth do have an impact on the consumption and, hence, marginal utility of active agents. Second, because the degree of market segmentation is endogenous, the size of this impact changes systematically with the level of money growth. In particular, as money growth increases, more agents choose to be active in financial markets, and the degree of risk due to market segmentation falls. With these two innovations, our model can deliver large and variable currency risk premia even though the fundamental shocks have constant variance.

Mechanically, our model generates variable risk premia because $\log c_A(\mu)$ is increasing and concave in $\log \mu$. To see the link between risk premia and $\log c_A(\mu)$, define $\phi(\mu)$ to be the elasticity of the marginal utility of active households to a change in money growth. With constant relative risk aversion preferences of the form $U(c) = c^{1-\sigma}/(1-\sigma)$, this elasticity is given by

$$(29) \qquad \phi(\mu) \equiv - \frac{d \log U'\big(c_A(\mu)\big)}{d \log \mu} = \sigma \frac{d \log c_A(\mu)}{d \log \mu}.$$

For later use, note that when $\log c_A(\mu)$ is increasing in $\log \mu, \phi(\mu) > 0$. The larger is $\phi(\mu)$, the more sensitive is the marginal utility of active households to money growth. Also note that when $\log c_A(\mu)$ is concave in $\log \mu, \phi(\mu)$ decreases in μ, so the marginal utility of active households is more sensitive to changes in money growth at low levels of money growth than at high levels. In this sense, the concavity implies that the variability of the pricing kernel decreases as money growth increases.

We now characterize features of our equilibrium in two propositions. In Proposition 2, we show that more households choose to become active as money growth and inflation increase. The result is intuitive: as inflation increases, so does the cost of not participating in the asset market, since the consumption of inactive households, namely, y/μ, falls as money growth μ increases. In Proposition 3, we show that, at least for low values of money growth, $\log c_A(\mu)$ is increasing and concave in $\log \mu$.

Proposition 2. As μ increases, more households become traders. In particular, $\bar{\gamma}'(\mu) > 0$ for

$\mu > 1$, and $\bar{\gamma}'(1) = 0$.

Proof. Differentiating equations (18) and (19) with respect to μ and solving for $\bar{\gamma}'$ gives

$$\bar{\gamma}'(\mu) = \frac{\left[U'\left(\frac{y}{\mu}\right) - U'(c_A)\right]\frac{y}{\mu} - U''(c_A)\left[c_A + \bar{\gamma} - (y/\mu)\right]\frac{1-F}{F}y/\mu^2}{U'(c_A) - U''(c_A)\left[c_A + \bar{\gamma} - (y/\mu)\right]f/F},$$

where to simplify we have omitted the arguments in the functions F,f,c_A, and $\bar{\gamma}$. Notice that $c_A(1) = y$ and $\bar{\gamma}(1) = 0$. Also note that (18) implies that if $\mu > 1$, then $c_A + \bar{\gamma} - y/\mu > 0$. To derive this result rewrite (18) as

$$c_A(\mu) + \frac{\int_0^{\bar{\gamma}(\mu)} \gamma f(\gamma)\, d\gamma}{F(\bar{\gamma}(\mu))} - y/\mu = \frac{y - y/\mu}{F(\bar{\gamma}(\mu))}, \tag{30}$$

use the inequality $\bar{\gamma}(\mu) \geq \left(\int_0^{\bar{\gamma}(\mu)} \gamma f(\gamma)\, d\gamma\right)/F(\bar{\gamma}(\mu))$, and note that the right side of (30) is strictly positive for $\mu > 1$. It follows from this result and (19) that $U'\left(\frac{y}{\mu}\right) - U'(c_A) > 0$ for $\mu > 1$. Finally, since U is strictly concave, $U''(c_A) < 0$; thus, $\bar{\gamma}' > 0$ for $\mu > 1$. Using similar results for $\mu = 1$, we get $\gamma'(1) = 0.Q.E.D.$

Proposition 3. The log of the consumption of active households $c_A(\mu)$ is strictly increasing and strictly concave in $\log \mu$ around $\mu = 1$. In particular, $\phi(1) > 0$ and $\phi'(1) < 0$.

Proof. We first show that $\phi(1) = \sigma[1 - F(0)]/F(0)$, which is positive when $F(0) > 0$. To see this, differentiate (18) with respect to μ and $\bar{\gamma}$, and use, from Proposition 2, that $\bar{\gamma}'(1) = \bar{\gamma}(1) = 0$, to get

$$c_A'(1) = y\frac{1 - F(0)}{F(0)}.$$

Using this expression for $c_A'(1)$ and using $c_A(1) = y$ in $\phi(1) = \sigma c_A'(1)/c_A(1)$ gives the answer.

We next show that $\phi'(1) = -\phi(1)/F(0)$, which is negative because $\phi(1) > 0$ and $F(0) > 0$. To see this, first differentiate (29) to get

$$\phi'(1) = \sigma\left[\frac{c_A''(1)}{c_A(1)} + \frac{c_A'(1)}{c_A(1)} - \left(\frac{c_A'(1)}{c_A(1)}\right)^2\right]. \tag{31}$$

Second, differentiate (18) with respect to μ and $\bar{\gamma}$ and use the result at $\mu = 1$, $\bar{\gamma}'(\mu) = \bar{\gamma}(\mu) = 0$, and $c_A(\mu) + \bar{\gamma}(\mu) - y/\mu = 0$ to get

$$c_A''(1) = -2y\frac{1 - F(0)}{F(0)}.$$

Using these expressions for c'_A and c''_A in (31), we obtain the desired result. *Q.E.D.*

In Proposition 2 we showed that more agents pay the fixed cost when money growth increases, and in Proposition 3 we showed that locally the consumption of active agents is increasing and concave in money growth.

A. A Numerical Example

Here we consider a simple numerical example that demonstrates these features more broadly. We interpret one time period as a month. We let $y = 1$,$\sigma = 2$, and for fixed costs we let fraction $F(0) = .125$ of agents have zero fixed costs and the remainder have fixed costs with a uniform distribution on $[0,b]$ with $b = .1$. In Figure 2 we plot $\log c_A(\mu)$ against $\log \mu$ (annualized). This figure shows that the consumption of active agents is increasing and concave in money growth in the relevant range. Because of this nonlinearity, even if the fundamental shocks—here, money growth rates—have constant conditional variances, the resulting pricing kernels have time-varying conditional variances.

To capture the nonlinearity of $c_A(\mu)$ in a tractable way when computing the asset prices implied by our model, we take a second-order approximation to the marginal utility of active households to get

$$(32) \quad \log U'\big(c_A(\mu_t)\big) = \log U'\big(c_A(\bar{\mu})\big) - \phi\hat{\mu}_t + \frac{1}{2}\eta\hat{\mu}_t^2,$$

where $\hat{\mu}_t = \log \mu_t - \log \bar{\mu}$,

$$(33) \quad \phi \equiv - \left.\frac{d \log U'\big(c_A(\mu)\big)}{d \log \mu}\right|_{\mu=\bar{\mu}} = \sigma \left.\frac{d \log c_A(\mu)}{d \log \mu}\right|_{\mu=\bar{\mu}}$$

$$\eta \equiv \left.\frac{d^2 \log U'\big(c_A(\mu)\big)}{(d \log \mu)^2}\right|_{\mu=\bar{\mu}} = -\sigma \left.\frac{d^2 \log c_A(\mu)}{(d \log \mu)^2}\right|_{\mu=\bar{\mu}}.$$

For our numerical example, $\phi = 10.9$ and $\eta = 1007$ when $\bar{\mu}$ is 5% at an annualized rate.

Motivated by our previous results, we assume that $\phi > 0$ and $\eta > 0$. With this parameterization, we have that the pricing kernel is given by

$$(34) \quad \log m_{t+1} = \log \beta/\bar{\mu} - (\phi + 1)\hat{\mu}_{t+1} + \frac{1}{2}\eta\hat{\mu}_{t+1}^2 + \phi\hat{\mu}_t - \frac{1}{2}\eta\hat{\mu}_t^2.$$

Throughout, we assume that the log of home money growth has normal innovations or shocks, so that

$$(35) \quad \hat{\mu}_{t+1} = E_t \hat{\mu}_{t+1} + \varepsilon_{t+1}$$

and likewise for foreign money growth, where ε_{t+1} and ε^*_{t+1} are the independent shocks across countries and are both normal with mean zero and variance σ^2_ε. For interest rates to be well-defined with our quadratic approximation, we need

$$(36) \quad \eta \sigma^2_\varepsilon < 1,$$

which we assume holds throughout.

5. Time-Varying Risk Premia

Now we use our pricing kernel (34) to show how the risk premium varies systematically with money growth. We show that the risk premium varies even if the shocks to money growth have constant conditional variances. In particular, we show that, locally, an increase in money growth decreases the risk premium p_t. We also give conditions under which the variation in the risk premium is large.

Recall that the risk premium can be written in terms of the pricing kernel as in (27):

$$(37) \quad p_t = \log E_t m_{t+1} - E_t \log m_{t+1} - (\log E_t m^*_{t+1} - E_t \log m^*_{t+1}).$$

Note that if the pricing kernel m_{t+1} were a conditionally lognormal variable, then, as is well-known, $\log E_t m_{t+1} = E_t \log m_{t+1} + (1/2)\mathrm{var}_t \log m_{t+1}$. In such a case the risk premium p_t would equal half the difference of the conditional variances of the log kernels. Given our approximation (34), however, the pricing kernels are not conditionally lognormal, but a similar relation between the risk premium and the conditional variances of the kernels holds, as we show in the next proposition (proved in Appendix B).

Proposition 4. Under (34), the risk premium is

$$(38) \quad p_t = \frac{1}{2} \frac{1}{(1 - \eta\sigma^2_\varepsilon)} \left(\mathrm{var}_t \log m_{t+1} - \mathrm{var}_t \log m^*_{t+1} \right),$$

where

$$(39) \quad \mathrm{var}_t(\log m_{t+1}) = \frac{1}{2} \frac{[-(1+\phi) + \eta E_t \hat{\mu}_{t+1}]^2 \sigma^2_\varepsilon}{1 - \eta\sigma^2_\varepsilon} + \frac{3}{4}\eta^2\sigma^4_\varepsilon$$

and a symmetric formula holds for $\text{var}_t\left(\log m^*_{t+1}\right)$. The expected depreciation of the exchange rate is given by

$$(40)\quad E_t \log e_{t+1} - \log e_t =$$

$$-(\phi+1)E_t(\hat{\mu}^*_{t+1} - \hat{\mu}_{t+1}) + \frac{1}{2}\eta E_t(\hat{\mu}^{*2}_{t+1} - \hat{\mu}^2_{t+1}) + \phi(\hat{\mu}^*_t - \hat{\mu}_t) - \frac{1}{2}\eta(\hat{\mu}^{*2}_t - \hat{\mu}^2_t).$$

To see how the risk premium varies with money growth, we calculate the derivative of the risk premium and evaluate it at $\mu_t = \bar{\mu}$ to get

$$(41)\quad \frac{dp_t}{d\hat{\mu}_t} = -\frac{\eta(\phi+1)\sigma^2_\varepsilon}{1-\eta\sigma^2_\varepsilon}\frac{dE_t\hat{\mu}_{t+1}}{d\hat{\mu}_t}.$$

Under (36), from (41) we know that the risk premium falls with home money growth if $\eta > 0$ and if money growth is persistent, in that $dE_t\hat{\mu}_{t+1}/d\hat{\mu}_t$ is positive.

The basic idea behind why the risk premium decreases with the money growth rate has two parts. First, since money growth is persistent, a high money growth rate in period t leads households to forecast a higher money growth rate in period $t+1$. Second, in any period, since η is positive, the marginal utility of active households is concave in the rate of money growth in that period. So as money growth increases, the sensitivity of marginal utility to fluctuations in money growth decreases. Thus, a high rate of money growth in period t leads households to predict that marginal utility in period $t+1$ will be less variable. Hence, the risk premium decreases with the money growth rate.

Next consider the variability of the risk premium. Expanding the terms in (39), we have that $\text{var}_t(\log m_{t+1})$ equals a constant plus

$$\frac{\eta\sigma^2_\varepsilon}{1-\eta\sigma^2_\varepsilon}\left[-(1+\phi)E_t\hat{\mu}_{t+1} + \frac{\eta}{2}(E_t\hat{\mu}_{t+1})^2\right].$$

As long as $E_t\hat{\mu}_{t+1}$ is approximately normal, so that the covariance between $E_t\hat{\mu}_{t+1}$ and $(E_t\hat{\mu}_{t+1})^2$ is approximately zero, the variability of the risk premium is increasing in ϕ,η, and σ^2_ε. The intuition for this result is the same as that for (41). As these parameters increase, the conditional variance of the pricing kernel changes more with a given change in the growth rate of money.

6. The Forward Premium Anomaly

In the data, high interest rate currencies are expected to appreciate. Here we show how the variations in the risk premium, interest rates, and exchange rates implied by our model allow it to generate this forward premium anomaly.

Recall that interest rates, exchange rates, and the risk premium are linked in our model by

$$(42) \quad i_t - i_t^* = E_t \log e_{t+1} - \log e_t - p_t.$$

For high interest rate currencies to be expected to appreciate, we need a shock to increase $i_t - i_t^*$ and decrease $E_t \log e_{t+1} - \log e_t$ (that is, to generate an expected appreciation). From (42), we know that the shock must also make the risk premium p_t fall. Our model produces this pattern as follows. As we have seen, a persistent increase in money growth leads the risk premium to fall. When this increase in money growth also leads to an expected appreciation smaller in magnitude than the fall in the risk premium, then the interest differential increases, and our model generates the forward premium anomaly.

The simplest case to study is when exchange rates are random walks, for then an increase in money growth has no effect on the expected appreciation. Here the covariance between the interest differentials and expected change in the exchange rate is zero, so the model generates, at least weakly, the forward premium anomaly. The more general case is when a persistent increase in money growth leads to a moderate expected appreciation.

We begin our study of the general case with a discussion of the link between money growth and expected changes in exchange rates, and then present a numerical example.

A. Exchange Rates and Money Growth

The real exchange rate $v_t = e_t P_t^* / P_t$ can help us explain our results. With symmetry between the two countries, we can easily show that the real exchange rate is given by

$$v_t = \frac{U'\big(c_A^*(\mu_t^*)\big)}{U'\big(c_A(\mu_t)\big)}.$$

We can then write the expected change in the nominal exchange rate as the sum of the expected change in the real exchange rate and the expected inflation differential:

$$(43) \quad E_t \log e_{t+1} - \log e_t = (E_t \log v_{t+1} - \log v_t) + E_t[\log(P_{t+1}/P_t) - \log(P_{t+1}^*/P_t^*)]$$

Using (20), (21), and (26) together with (15) and its foreign analog, we can write the right side of (43) as

$$(44)\quad E_t[\log U'(c^*_{At+1})/U'(c_{At+1}) - \log U'(c^*_{At})/U'(c_{At})] + E_t[\log \mu_{t+1} - \log \mu^*_{t+1}],$$

where the first bracketed term corresponds to the change in the real exchange rate and the second to the expected inflation differential.

Hence, we can decompose the effect of money growth changes on the expected change in the nominal exchange rate into two parts: a *market segmentation effect* and an *expected inflation effect*. The market segmentation effect measures the impact of an increase in money growth on the expected change in the real exchange rate through its impact on the marginal utilities in the first term in (44). This effect is not present in the standard model, which has no segmentation. The expected inflation effect measures the impact of an increase in money growth on the expected inflation differential in the second term in (44).

Now consider the impact of a persistent increase in money growth on the expected change in the nominal exchange rate. The expected inflation effect is simply

$$(45)\quad d(E_t \log \mu_{t+1})/d \log \mu_t.$$

This effect is larger the more persistent is money growth. In the standard model, this is the only effect, so that an increase in money growth of one percentage point leads to an expected nominal depreciation of size $d(E_t \log \mu_{t+1})/d \log \mu_t$.

The size of the market segmentation effect depends on both the degree of market segmentation and the persistence of money growth. A persistent increase in the home money growth rate μ_t affects both the current real exchange rate

$$(46)\quad \log v_t = \log U'\big(c^*_A(\mu^*_t)\big)/U'\big(c_A(\mu_t)\big)$$

and, by increasing the expected money growth rate in $t+1$, the expected real exchange rate

$$(47)\quad E_t \log v_{t+1} = E_t \log U'\big(c^*_A(\mu^*_{t+1})\big)/U'\big(c_A(\mu_{t+1})\big).$$

To understand the market segmentation effect, suppose first that money growth is not persistent, but rather independently and identically distributed. Then changes in home money growth affect only the current real exchange rate. A money growth increase increases the consumption of the home active households in the current period, thus decreasing both their marginal utility and the current real exchange rate

$$\log v_t = \log U'\big(c_A^*(\mu_t^*)\big)/U'\big(c_A(\mu_t)\big).$$

Since the expected real exchange rate in $t+1$ is unchanged by the money growth shock, the real exchange rate is expected to appreciate from t to $t+1$; that is, $E_t \log v_{t+1} - \log v_t$ falls. The magnitude of this effect is larger the greater is the degree of market segmentation, as measured by $\phi(\mu_t)$.

Now suppose that money growth is persistent. Then changes in the money growth rate in period t also affect the expected money growth rate in $t+1$ and thus the expected real exchange rate in period $t+1$ as well. The effect of money growth on the expected real exchange rate depends on both the degree of market segmentation and the persistence of money growth, as measured by $d(E_t\hat{\mu}_{t+1})/d\hat{\mu}_t$. The pricing kernel (34) implies that

$$E_t \log v_{t+1} - \log v_t =$$

$$\phi(E_t\hat{\mu}_{t+1} - \hat{\mu}_t) - \frac{\eta}{2}[(E_t\hat{\mu}_{t+1})^2 - \hat{\mu}_t^2] - \phi(E_t\hat{\mu}_{t+1}^* - \hat{\mu}_t^*) + \frac{\eta}{2}[(E_t\hat{\mu}_{t+1}^*)^2 - \hat{\mu}_t^{*2}].$$

Hence, an increase in the home money growth rate $\hat{\mu}_t$ leads to an expected change in the real exchange rate of

$$(48) \quad \frac{d}{d\hat{\mu}_t}(E_t \log v_{t+1} - \log v_t) = \phi\left[\frac{d(E_t\hat{\mu}_{t+1})}{d\hat{\mu}_t} - 1\right],$$

where we have evaluated this derivative at $\hat{\mu}_t = 0$. As long as money growth is mean-reverting, in that $d(E_t\hat{\mu}_{t+1})/d\hat{\mu}_t < 1$, an increase in money growth near the steady state leads to an expected real appreciation. Clearly, the magnitude of the expected real appreciation depends on both the degree of market segmentation, as measured by ϕ, and the degree of persistence in money growth, as measured by $d(E_t\hat{\mu}_{t+1})/d\hat{\mu}_t$.

Note that the market segmentation effect and the expected inflation effect have opposite signs. If the market segmentation effect dominates, then locally an increase in home money growth leads to an expected appreciation of the nominal exchange rate. This will occur when

$$(49) \quad \frac{d(E_t\hat{\mu}_{t+1})}{d\hat{\mu}_t} \leq \frac{\phi}{1+\phi}.$$

For this condition to hold, markets must be sufficiently segmented relative to the persistence of money growth. If (49) holds as an equality, then the market segmentation effect exactly cancels the expected inflation effect, and the nominal exchange rate will be locally a random walk.

B. A Numerical Example

Here we use a simple numerical example to illustrate the type of interest rate and exchange rate behavior that our model can generate. We have constructed this example so that the exchange rate is a martingale. Hence, interest rates are driven entirely by movements in the risk premium and the slope coefficient in the Fama regression is zero. Interestingly, the example has some qualitative properties that are similar to the data: interest differentials are persistent, and the exchange rate is an order of magnitude more volatile than interest differentials. We think of this example as illustrating some of the behavior our model can generate, rather than being a definitive quantitative analysis of the properties of interest rates and exchange rates.

In this example we choose the processes for the money growth rates to ensure that the nominal exchange rate follows a random walk. Specifically, we choose these processes so that

$$E_t \log e_{t+1} - \log e_t = E_t(\log m^*_{t+1} - \log m_{t+1}) = 0.$$

Since the pricing kernel in each country is a function of only that country's money growth, we choose these processes so that for the home country $E_t \log m_{t+1} = \log \beta/\bar{\mu}$, where $\log m_{t+1}$ is given by (34); we do likewise for the foreign country. Because of the nonlinearity of our pricing kernels, we must choose a nonlinear process for the money growth process to make exchange rates a random walk. For the home and foreign countries we let these *baseline processes* be of the form

$$(50) \quad \hat{\mu}_{t+1} = g(\hat{\mu}_t) + \varepsilon_{t+1},\ \hat{\mu}^*_{t+1} = g(\hat{\mu}^*_t) + \varepsilon^*_{t+1}.$$

We choose $g(\mu)$ to be the stable root of the resulting quadratic equation that results from substituting (50) into (40) and setting this expected depreciation to zero. We let ε_t and ε^*_t both be normal with mean zero and standard deviation σ_ε and with the correlation of ε_t and ε^*_t equal to ρ_ε.

We use the parameters $\phi = 10$ and $\eta = 1,000$. We think of these parameters as round numbers that are motivated by those in our earlier example. As before, we assume that one period in the model is a month, and we set $\bar{\mu}$ corresponding to an annualized inflation rate of 5%. We set $\sigma_\varepsilon = .0035$ and ρ_ε equal to 1/2. With these parameters the resulting money growth process of the form (50) is similar to that of an AR1 process with a serial correlation of .90. To demonstrate this similarity, in Figure 3 we plot 245 realizations of our baseline money growth process (50) and this AR1 process using the same driving shocks ε_t.

In Table 1 we report on some properties of exchange rates and interest rates implied by this example and provide some similar statistics from the data. The statistics in the model are computed as the mean over 100,000 draws of length 245, while those in the data are averages of the statistics for seven European countries presented in Backus, Foresi, and Telmer (2001), each of which has 245 months of data. As the table demonstrates, in the data changes in the exchange rate are an order of magnitude more volatile than interest differentials. Also, changes in the exchange rate have virtually no serial correlation, whereas interest differentials are highly serially correlated. At a qualitative level our model reproduces these features of the data.

Table 1

Properties of Exchange Rates and Interest Rates

Statistics	Data	Model
Standard Deviations		
$\log e_{t+1} - \log e_t$	36	58
$i_t - i_t^*$	3.5	1.3
Autocorrelations		
$\log e_{t+1} - \log e_t$	.04	0
$i_t - i_t^*$	.83	.92

In our model in an infinite sample the slope coefficient in the Fama regression would be zero. We were also interested in what our model implies for this coefficient for samples of the length used in the data to estimate this coefficient. Figure 4 is a histogram of 1,000 estimates of the slope coefficients of the Fama regression from simulated samples of the length 245, which is the length used in Backus, Foresi, and Telmer (2001). As is evident, our model is consistent with having a wide variety of slope coefficients in small samples, including very negative ones. In addition, it is worth noting that the mean value for the slope coefficient across the 100,000 draws is -3.69, which is substantially lower than its population value of zero, indicating the presence of significant small sample bias.

C. Some Cross-Section Implications

So far we have focused on the time series implications of our model. Now we discuss some cross-section implications. A key mechanism at work in our market segmentation model is that as the money growth rate rises, so does the inflation rate; thus, gains from participating in the asset market rise with money growth. As these gains rise, more households choose to be active, and the amount of risk in the economy falls. In an economy with a high enough mean inflation rate, then, risk in the asset market is sufficiently low that the forward premium anomaly disappears. Our model thus implies that the market segmentation effect is smaller in countries with higher inflation rates.

More precisely, if the distribution of fixed costs is bounded and the risk aversion parameter $\sigma > 1$, then clearly, there is some sufficiently high inflation rate such that for all rates higher than that, all households are active and consumption is constant. Then the model reduces to a standard one similar to that of Lucas (1982), in which risk premia are constant and there is no forward premium anomaly.

Some evidence for this cross-section implication has been found by Bansal and Dahlquist (2000). They study a data set for 28 emerging and developed countries and find that the forward premium anomaly is mostly present in the developed countries and mostly absent in the emerging countries. In regressions for their entire data set, Bansal and Dahlquist find that countries with higher inflation rates tend to have smaller forward premium anomalies.

7. Conclusion

We have constructed a simple general equilibrium model with endogenously segmented asset markets and have shown that this sort of friction is a potentially important part of a complete model of exchange rates. Relative to the existing literature, we make several contributions.

We show that making market segmentation endogenous can lead to risk driving most of the movements in interest differentials, as it does in the data. Specifically, the model implies that the market segmentation effects—and, hence, risk—varies systematically with the level of inflation. In the time series, this feature implies that exchange rate risk varies systematically with inflation. In the cross section, it implies that the market segmentation effect is less important in high inflation countries.

Appendix A: Proof of Uniqueness

We show that equations (18) and (19) have at most one solution for any given μ. To show this result, solve for $\bar{\gamma}$ as a function of c_A from (19) and suppress explicit dependence of μ to get

$$\bar{\gamma}(c_A) = \frac{U(c_A) - U(y/\mu)}{U'(c_A)} - [c_A - y/\mu] .$$

Note that

$$\frac{d\bar{\gamma}(c_A)}{dc_A} = -\frac{U''(c_A)}{(U'(c_A))^2} (U(c_A) - U(y/\mu)) . \tag{51}$$

Using (19) it follows that $d\bar{\gamma}(c_A)/dc_A$ is positive when $(c_A + \bar{\gamma} - y/\mu) > 0$ and $d\bar{\gamma}(c_A)/dc_A$ is negative when $(c_A + \bar{\gamma} - y/\mu) < 0$. Substituting $\bar{\gamma}(c_A)$ into (18) and differentiating the left side of the resulting expression with respect to c_A gives

$$F(\bar{\gamma}(c_A)) + [c_A + \bar{\gamma}(c_A) - y/\mu] \frac{d\bar{\gamma}(c_A)}{dc_A}. \tag{52}$$

Using (51), we see that (52) is strictly positive and hence there is at most one solution to these equations.

Appendix B: Proof of Proposition 4

To prove Proposition 4 we derive two equations, (38) and (39).

To derive (38) start with (37). Compute $E_t \log m_{t+1}$ from (34). To compute $\log E_t m_{t+1}$ we must compute

$$\log E_t \exp\left(\left[-(\phi+1) + \eta E_t \hat{\mu}_{t+1}\right] \varepsilon_{t+1} + \frac{\eta}{2}\varepsilon_{t+1}^2\right).$$

To do so we use the result that if x is normally distributed with mean zero and variance σ^2 and satisfies $1 - 2b\sigma^2 > 0$, then

$$E \exp\left(ax + bx^2\right) = \exp\left(\frac{1}{2}\frac{a^2\sigma^2}{(1-2b\sigma^2)}\right)\left(\frac{1}{1-2\sigma^2 b}\right)^{1/2}. \tag{53}$$

To derive (53), note that

$$E \exp\left(ax + bx^2\right) = \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(ax + bx^2\right) \exp\left(-\frac{x^2}{2\sigma^2}\right) dx =$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(\frac{1}{2\sigma^2}\left[2\sigma^2ax+\left(2\sigma^2b-1\right)x^2\right]\right)dx=$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(\frac{1}{2\sigma^2}\left[-\left(1-2\sigma^2b\right)x^2+2\sigma^2ax-\left(\frac{\sigma^4a^2}{1-2\sigma^2b}\right)+\left(\frac{\sigma^4a^2}{1-2\sigma^2b}\right)\right]\right)dx=$$

$$\exp\left(\frac{1}{2}\frac{a^2\sigma^2}{1-2b\sigma^2}\right)\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(-\frac{1}{2\sigma^2}\left[\left(1-2\sigma^2b\right)^{1/2}x-\frac{\sigma^2a}{(1-2\sigma^2b)^{1/2}}\right]^2\right)dx=$$

$$\exp\left(\frac{1}{2}\frac{a^2\sigma^2}{1-2b\sigma^2}\right)\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(-\frac{(1-2\sigma^2b)}{2\sigma^2}\left[x-\frac{\sigma^2a}{(1-2\sigma^2b)}\right]^2\right)dx,$$

which equals (53).

We can derive (39) using (34) together with the standard results that $E_t\varepsilon_{t+1}^4=3\sigma_\varepsilon^4$ and $E_t\varepsilon_{t+1}^3=0$.

Notes

[1]This anomaly can also be stated in terms of forward exchange rates. To see this, note that the forward exchange rate f_t is the price specified in a contract in t in which the buyer has the obligation to transfer f_t dollars in $t+1$ and receive one euro. The forward premium is the forward rate relative to the spot rate f_t/e_t. Arbitrage implies that $\log f_t - \log e_t = i_t - i_t^*$. Thus, (3) can be restated as $\text{cov}(\log f_t - \log e_t, \log e_{t+1} - \log e_t) < 0$. There is, then, a tendency for the forward premium and the expected change in exchange rates to move in opposite directions. This observation contradicts the hypothesis that the forward rate is a good predictor of the future exchange rate.

[2]Variants of this model can also be considered in which the fixed cost for each household varies randomly over time. As will be clear from what follows, for the appropriate set of sufficient conditions, the cash-in-advance constraints would always bind in those variants, and the equilibrium would be identical.

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Figure 1 **Timing in the Two Markets**

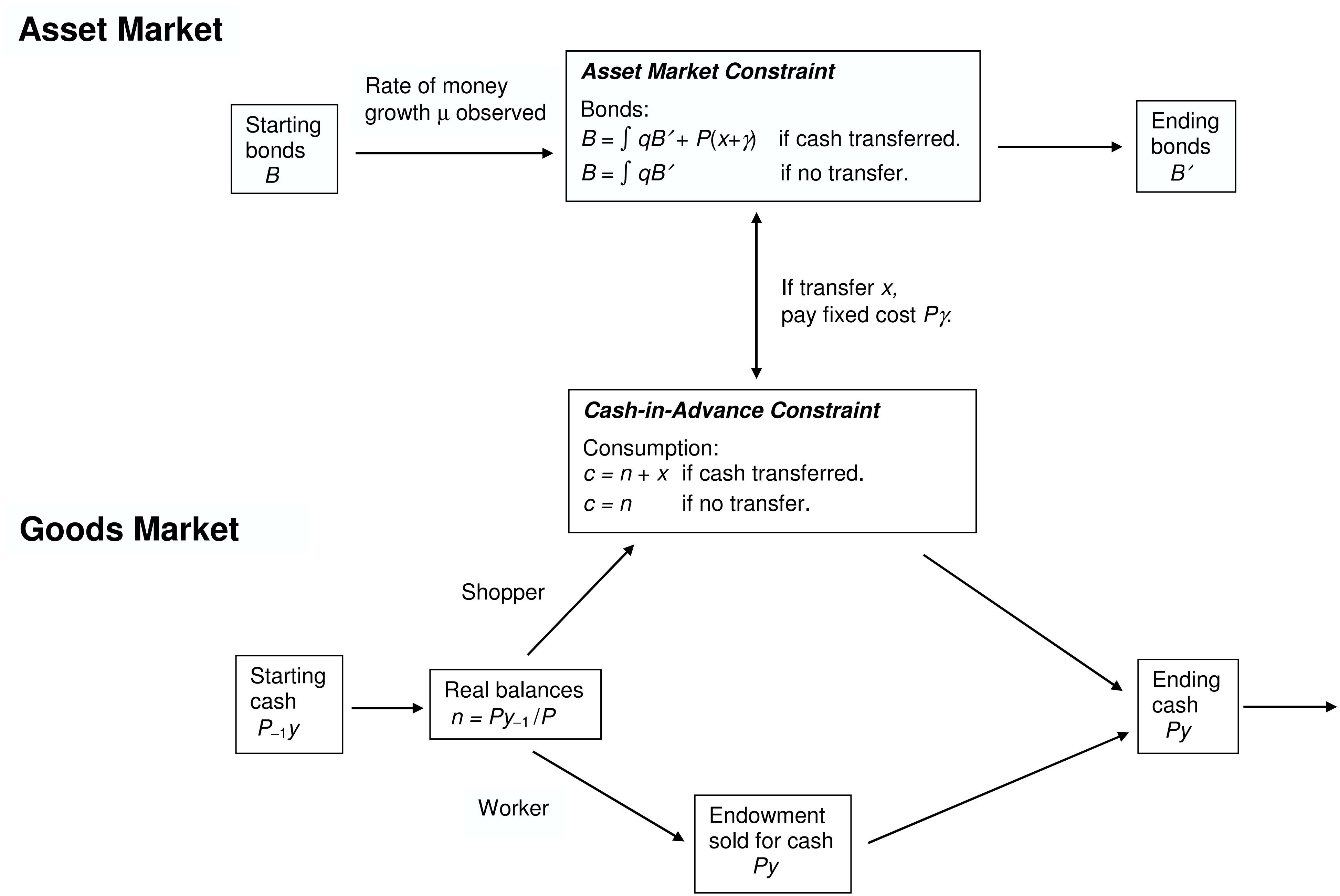

Figure 2 **The Log of Consumption of Active Households**

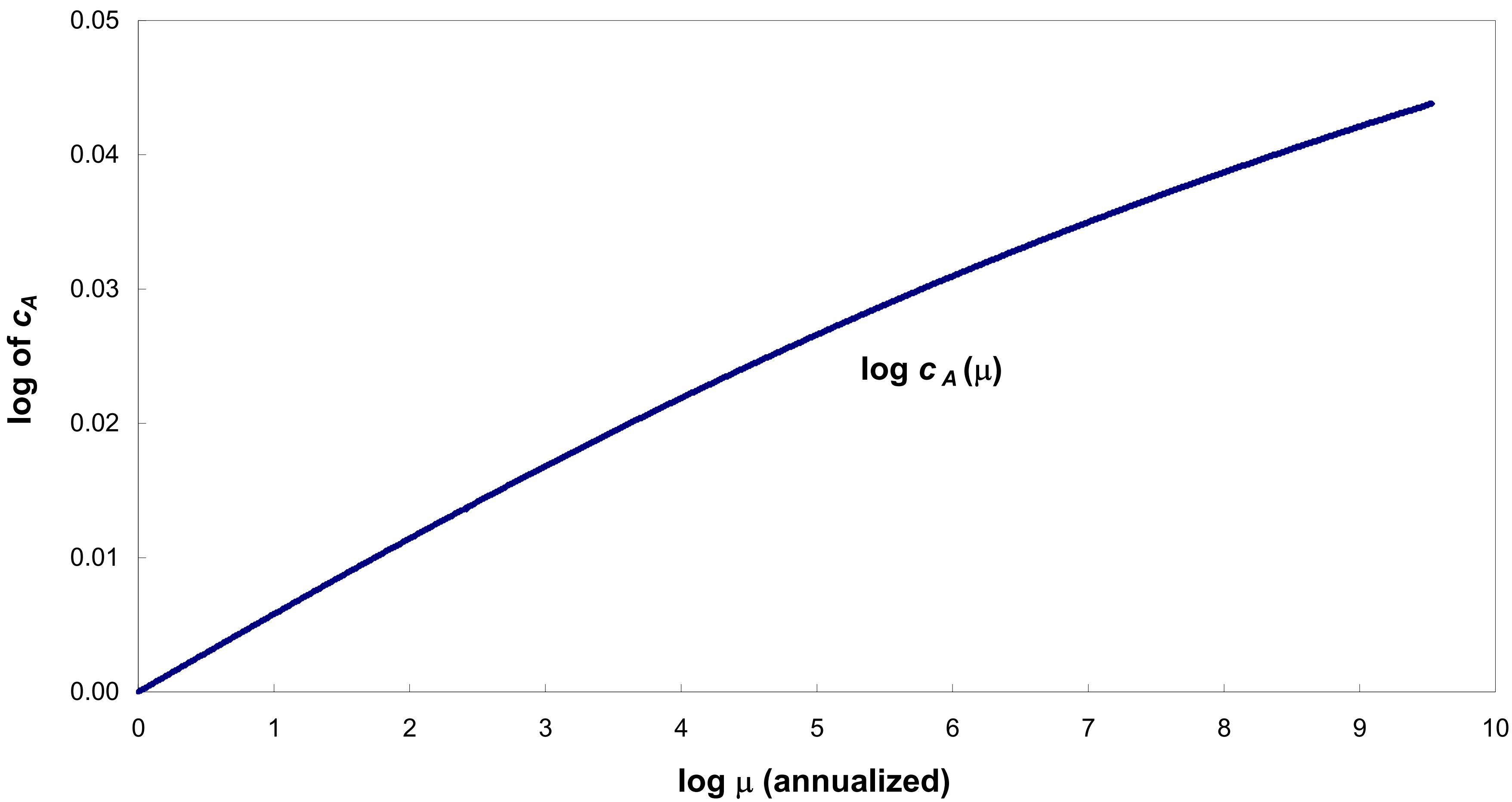

Figure 3 **Realizations of Money Growth Using Our Baseline Process and an AR1 Process**

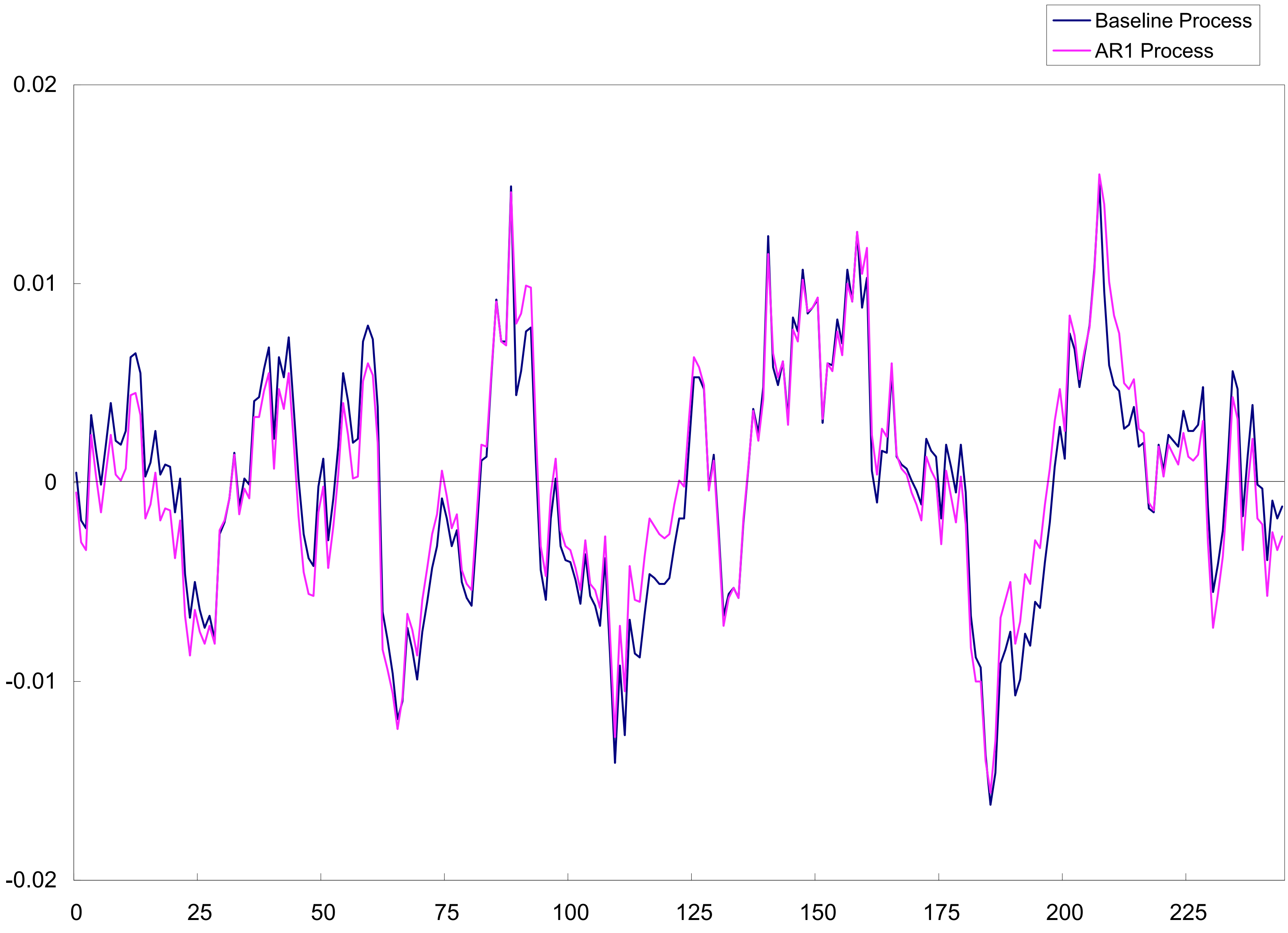

Figure 4 **Histogram of the Slope Coefficients of the Fama Regression**