Crises and Prices
Information Aggregation, Multiplicity and Volatility[*]

George-Mario Angeletos
MIT and NBER
`angelet@mit.edu`

Iván Werning
MIT, NBER and UTDT
`iwerning@mit.edu`

First draft: January 2004
This version: December 2004

Abstract

Crises, such as currency attacks, bank runs and riots, can be described as times of non-fundamental volatility. We argue that crises are also times when endogenous sources of information are closely monitored and thus an important part of the phenomena. We study the role of endogenous information in generating non-fundamental volatility by introducing a financial market in a coordination game where agents have heterogeneous information about the fundamental. The equilibrium price aggregates information without restoring common knowledge. In contrast to the case with exogenous information, we find that uniqueness may not be obtained as a perturbation from common knowledge: multiplicity is ensured when individuals observe fundamentals with small idiosyncratic noise. Interestingly, multiplicity may also emerge in the financial price. Finally, when the equilibrium is unique it becomes more sensitive to non-fundamental shocks as private noise is reduced.

JEL Codes: D8, E5, F3, G1.

Keywords: Multiple equilibria, coordination, global games, speculative attacks, currency crises, bank runs, financial crashes, rational expectations.

---

[*] For useful comments and suggestions, we thank Daron Acemoglu, Fernando Alvarez, Manuel Amador, Gadi Barlevy, V.V. Chari, Christian Hellwig, Patrick Kehoe, Jose Scheinkman, Balazs Szentes, Aleh Tsyvinski and seminar participants at MIT, Stanford, UCLA, UC Berkeley, Duke, the Minneapolis and Chicago Federal Reserve Banks, the 2004 Minnesota Summer Workshop in Macroeconomic Theory, and the 2004 UTDT Summer Workshop in International Economics and Finance.

1 Introduction

It's a love-hate relationship, economists are at once fascinated and uncomfortable with multiple equilibria. On the one hand, economic and political crises involve large and abrupt changes in outcomes, but often lack obvious comparable changes in fundamentals. Many attribute an important role to more or less arbitrary shifts in 'market sentiments' or 'animal spirits', and models with multiple equilibria formalize these ideas. On the other hand, models with multiple equilibria can also be viewed as incomplete theories, which ultimately should be extended along some dimension to resolve the indeterminacy.

The first view is represented by a large empirical and theoretical literature. On the empirical side, Kaminsky (1999), for example, documents that the likelihood of a crisis is affected by observable fundamentals, but that a significant amount of volatility remains unexplained – crises are largely unpredictable. On the theoretical side, models featuring multiple equilibria attempt to address such non-fundamental volatility. Bank runs, currency attacks, debt crises, financial crashes, riots and political regime changes are modeled as a coordination game: attacking a regime – such as a currency peg or the banking system – is worthwhile if and only if enough agents are also expected to attack.[1]

Morris and Shin (1998, 2000, 2003), building on Carlson and van Damme (1993), contribute to the second view by showing that a unique equilibrium survives in such coordination games when individuals observe fundamentals with small enough private noise. The result is most striking when seen as a perturbation around the original common-knowledge model, which is ridden with equilibria. As the noise in private information vanishes, agents become perfectly informed, yet the equilibrium outcome is uniquely determined. Most importantly, their contribution highlights the importance of the information structure in environments with complementarities.

The aim of this paper is to understand the role of information in crises. We focus on two distinct forms of non-fundamental volatility: the existence of multiple equilibria and the sensitivity of a unique equilibrium to non-fundamental disturbances. We argue that endogenizing public information is crucial for these questions.

Information is typically taken as exogenous in coordination models, but is largely endogenous in most situations of interest. Financial prices and macroeconomic indicators convey information regarding others' actions and their beliefs about the underlying fundamentals. Such indicators are monitored intensely during times of crises and appear to be an important part of the phenomena. As an example, consider the Argentine 2001-2002 crisis, which included devaluation of the peso, sovereign-debt default, and suspension of bank payments.

---

[1] See, for example, Diamond and Dybvig (1983), Obstfeld (1986, 1996), Velasco (1996), Calvo (1988), Cooper and John (1988), Cole and Kehoe (1996). Cooper (1998) provides an excellent review.

Leading up to the crisis throughout 2001, the peso-forward rate and bank deposits deteriorated steadily. These variables, and others, were widely reported by news media and investor reports, and closely watched by people making crucial economic decisions.

These observations lead us to introduce endogenous sources of public information in a coordination game. In our baseline model, individuals observe their private signals and the price of a financial asset, whose dividend depends on the underlying fundamentals or the outcome of the coordination game. The rational-expectations equilibrium price aggregates disperse private information, but avoids perfect revelation due to noise in the aggregation process, as in Grossman and Stiglitz (1976).[2]

The main insight to emerge is that the precision of *endogenous* public information increases with the precision of *exogenous* private information. When private signals are more precise, individuals' asset demands are more sensitive to their information. As a result, the equilibrium price reacts relatively more to fundamental than to non-fundamental variables, thus conveying more precise public information.

This result has important implications for the determinacy of equilibria, as a horse-race between private and public information emerges. The direct effect of an increase in the precision of private information is that individuals find it harder to coordinate, as each relies more on her own distinct information. However, the resulting increase in the precision of endogenous public information facilitates coordination by helping individuals better forecast each others' actions. This indirect effect typically dominates and reverses the limit result: multiplicity is *ensured* when individuals observe fundamentals with small enough private noise.

Uniqueness therefore cannot be attained as a small perturbation around common knowledge. To illustrate this point, Figure 1 displays the regions of uniqueness and multiplicity in the space of exogenous levels of public and private noise, σ_ε and σ_x, respectively. Multiplicity is ensured when either noise is sufficiently small. In this sense, public and private noise have symmetric effects.[3]

Interestingly, multiplicity may emerge not only in the regime outcome but also in the asset price. This occurs when the asset's dividend depends on the coordination game. Multiple equilibrium prices are then sustained by different self-fulfilling expectations about the dividend, which in turn are facilitated by the information conveyed by the price.

In regions where the equilibrium is unique, we perform comparative statics. We find that

---

[2] Atkeson (2000) first pointed out that perfectly-revealing asset markets could restore common knowledge. By introducing noise in the aggregation process, we ensure that none of our results are driven by restoring common knowledge.

[3] Moreover, equilibria outcomes are continuous with respect to the exogenous noise parameters: we show that the common-knowledge outcomes are obtained in the limit as either $\sigma_x \to 0$ or $\sigma_\varepsilon \to 0$. In contrast, when information is exogenous, public noise contributes to uniqueness, while private noise contributes to multiplicity; and the Morris-Shin limit result discussed earlier illustrates a sharp discontinuity around zero private noise.

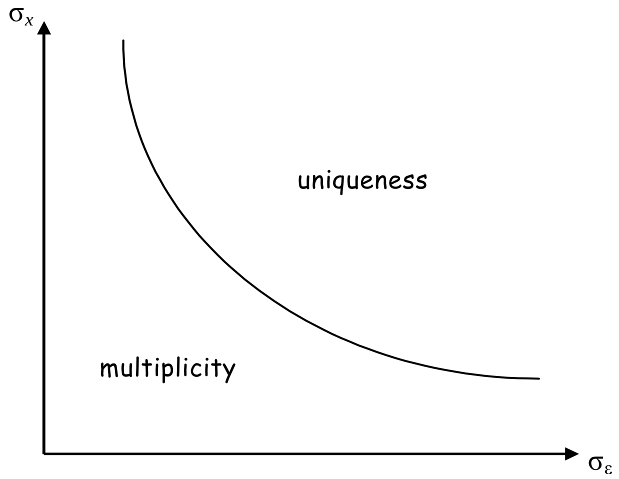

Figure 1: σ_x measures the exogenous noise in private information and σ_ε the exogenous public noise in the aggregation of information.

a reduction in exogenous noise, by helping agents better align their choices, may increase the sensitivity of the regime outcome to non-fundamental disturbances. When the dividend is endogenous this may also increase volatility in the financial price. Thus, lower noise may increases volatility by either introducing multiplicity or making the unique equilibrium more sensitive to non-fundamental shocks.

Finally, motivated by bank runs and riots, in Section 5 we also consider a model in which individuals do not trade a financial asset but instead observe a direct noisy public signal about others' actions. This model allows us to introduce endogenous public information into the Morris-Shin framework with minimal modifications. This model also brings a main element of herding models – observing others' actions – into coordination games. Our main results regarding equilibrium multiplicity carry over here, illustrating that the key is information aggregation, not the particular form of it.

Our main interest is the role of information and coordination in non-fundamental volatility. Chari and Kehoe (2003) also focus on non-fundamental volatility as the distinct feature of crises, but within the context of a herding model. Indeed, the model in Section 5, which allows individuals to observe signals of others actions, brings coordination games and herding models closer together.

Our analysis builds on Morris and Shin, underscoring their general theme that the information structure is crucial in coordination games. Our focus is on endogenous sources of public information, such as financial prices and other macroeconomic indicators that aggregate private information dispersed in the economy. Angeletos, Hellwig and Pavan (2003, 2004) also endogenize information, but in different ways: they examine the signaling and coordinating effects of policy, and the dynamics of information in a dynamic global game.

Most related is Hellwig, Mukherji and Tsyvinski (2004) who consider a currency-crises model in which financial prices directly affect the coordination outcome. In particular, they focus on how the determinacy of equilibria depends on whether the central bank's decision to devalue is triggered by large reserve losses or high interest rates. In their model, multiple equilibria also survive for small levels of noise.

Tarashev (2003) also endogenizes interest rates in a currency-crises model, but in a model where the equilibrium is always unique. Dasgupta (2002) introduces signals of others' actions in an investment game, but assumes that these signals are entirely private, instead of public as in our paper.[4]

Finally, the paper contributes to the rational-expectations literature (e.g., Grossman and Stiglitz, 1976; Grossman, 1981) by introducing a coordinating role for financial prices. In this literature the payoff of an agent is typically independent of the actions of other agents for any given price. Consequently, the equilibrium price only provides information regarding the exogenous dividend. In contrast, in our framework the price also helps agents to predict each others' actions and align their non-financial choices. This novel coordinating role is crucial for our results regarding price multiplicity and volatility.[5]

Section 2 introduces the basic model and reviews the exogenous information benchmark. Section 3 incorporates a financial market and examines the determinacy of equilibria. Section 4 considers the comparative statics in regions where a unique equilibrium obtains. Section 5 studies the model with direct signals of others' actions. Section 6 concludes.

2 The Basic Model: Exogenous Information

Before introducing financial prices or other endogenous public signals, we review the basic model with exogenous information, as in Morris and Shin (1999, 2000).

Actions and Payoffs. There is a status quo and a measure-one continuum of agents, indexed by $i \in [0, 1]$. Each agent i can choose between two actions, either attack the status quo ($a_i = 1$) or not attack ($a_i = 1$). The payoff from not attacking is normalized to zero. The payoff from attacking is $1 - c > 0$ if the status quo is abandoned and $-c$ otherwise, where $c \in (0, 1)$ parametrizes the cost of attacking. The status quo, in turn, is abandoned if and

---

[4] Morris and Shin (2002) and Angeletos and Pavan (2004), on the other hand, consider the volatility and welfare effects of exogenous public information in a class of models where complementarities are weak enough that the equilibrium is always unique.

[5] Barlevy and Veronessi (2004) consider a Grossman-Stiglitz environment that admits multiple equilibria, but the source of multiplicity there is entirely different from ours. In their model, the dividend is exogenous, so the price does not play a coordinating role. Instead, multiplicity emerges from the non-linearity of the inference problem faced by uninformed traders when interacting with informed and less risk-averse agents.

only if $A > \theta$, where A denotes the mass of agents attacking and θ represents the exogenous fundamentals, namely the strength of the status quo. It follows that the payoff of agent i is

$$U(a_i, A, \theta) = a_i(R\left(A, \theta\right) - c) \tag{1}$$

where $R\left(A, \theta\right)$ denotes the regime outcome, with $R = 1$ if $A > \theta$ and $R = 0$ otherwise.

Interpretations. In models of self-fulfilling currency crises (e.g., Obstfeld, 1986, 1996; Morris and Shin, 1998), a "regime change" occurs when a sufficiently large mass of speculators attacks the currency, forcing the central bank to abandon the peg; θ then parametrizes the amount of foreign reserves or more generally the ability and willingness of the central bank to maintain the peg. In the context of bank runs (e.g., Goldstein and Pauzner; 2000; Rochet and Vives 2004), on the other hand, a "regime change" occurs once a sufficiently large number of depositors decide to withdraw their deposits, forcing the banking system to suspend payments.[6] Alternatively, one could interpret the model as an economy with investment complementarities (e.g., Cooper and John, 1988, Dasgupta, 2002).

In short, the key property of the payoff structure is a coordination motive: $U\left(1, A, \theta\right) - U\left(0, A, \theta\right)$ increases with A, meaning that the individual incentive to take one action increases with the mass of other agents taking the same action. If θ were commonly observed by all agents, both $A = 1$ and $A = 0$ would be an equilibrium whenever $\theta \in (\underline{\theta}, \overline{\theta}] \equiv (0, 1]$. The interval $(\underline{\theta}, \overline{\theta}]$ thus represents the set of "critical fundamentals" for which agents can coordinate on multiple courses of action under common knowledge.

Information. Following Morris-Shin, we assume θ is not common knowledge. In the beginning of the game, nature draws θ from a given distribution, which constitutes the agents' common prior about θ. For simplicity, this prior is taken to be the improper uniform over the entire real line. Agent i then receives a private signal $x_i = \theta + \sigma_x \xi_i$, where $\sigma_x > 0$ and $\xi_i \sim \mathcal{N}(0, 1)$ is idiosyncratic noise, i.i.d. across agents and independent of θ. Agents also observe an *exogenous* public signal $z = \theta + \sigma_z v$, where $\sigma_z > 0$ and $v \sim \mathcal{N}(0, 1)$ is common noise, independent of both θ and ξ.[7] The information structure is thus parametrized by the standard deviations σ_x and σ_z; or, equivalently, by $\alpha_x = \sigma_x^{-2}$ and $\alpha_z = \sigma_z^{-2}$, the *precisions* of private and public information.

Equilibrium Analysis. We focus on *monotone equilibria*, that is, Bayes-Nash equilibria such that, given a realization z of the public signal, an agent attacks if and only if the realization x of his private signal is less than a threshold $x^*(z)$.[8] The aggregate size of the

---

[6] Related interpretations include debt crises, financial crashes, and riots (Morris and Shin, 2004a, 2004b; Corsetti, Guimaraes, and Roubini, 2004; Atkeson, 2000).

[7] The analysis of the effects of public information is intractable without the Normality assumptions (see also Morris and Shin, 1999, 2000, 2003).

[8] The focus on monotone equilibria is without serious loss of generality for two reasons. First, when the

attack is then $A(\theta, z) = \Phi\left(\sqrt{\alpha_x}\left(x^*(z) - \theta\right)\right)$, implying that the status quo is abandoned if and only if $\theta \leq \theta^*(z)$, where $\theta^*(z)$ solves $A(\theta, z) = \theta$, or equivalently

$$x^*(z) = \theta^*(z) + \tfrac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*(z)). \tag{2}$$

It follows that the expected payoff from attacking is $\Pr\left[\theta \leq \theta^*(z)|x, z\right] - c$ and therefore $x^*(z)$ must solve the indifference condition $\Pr\left[\theta \leq \theta^*(z)|x, z\right] = c$. Since the posterior of the agent about θ is Normal with mean $\frac{\alpha_x}{\alpha_x+\alpha_z}x + \frac{\alpha_z}{\alpha_x+\alpha_z}z$ and precision $\alpha = \alpha_x + \alpha_z$, the indifference condition is

$$\Phi\left(\sqrt{\alpha_x + \alpha_z}\left(\theta^*(z) - \tfrac{\alpha_x}{\alpha_x+\alpha_z}x^*(z) - \tfrac{\alpha_z}{\alpha_x+\alpha_z}z\right)\right) = c. \tag{3}$$

Substituting (2) into (3) results in a single equation in $\theta^*(z)$:

$$\Phi^{-1}\left(\theta^*\left(z\right)\right) - \tfrac{\alpha_z}{\sqrt{\alpha_x}}\theta^*\left(z\right) = \sqrt{1 + \tfrac{\alpha_z}{\alpha_x}}\Phi^{-1}\left(1 - c\right) - \tfrac{\alpha_z}{\sqrt{\alpha_x}}z \tag{4}$$

Hence, an equilibrium is simply identified with a solution to (4) for $\theta^*\left(z\right)$, and $x^*\left(z\right)$ is then given by (2). The Appendix proves that a solution to (4) always exists and is unique for all z if and only if $\sigma_x \leq \sigma_z^2\sqrt{2\pi}$.

Proposition 1 (Morris-Shin) *In the game with exogenous information, the equilibrium is unique if and only if* $\sigma_x \leq \sigma_z^2\sqrt{2\pi}$.

Figure 2 depicts the regions of (σ_x, σ_z) for which the equilibrium is unique. For any $\sigma_z > 0$, uniqueness in ensured by $\sigma_x > 0$ sufficiently small. Moreover, the equilibrium dependence on the non-fundamental shock ε vanishes as $\sigma_x \to 0$.

Corollary 1 *In the limit as* $\sigma_x \to 0$, *there is a unique equilibrium in which* $R\left(\theta, z\right) \to 1$ *if* $\theta < \hat{\theta}$ *and* $R\left(\theta, z\right) \to 0$ *if* $\theta > \hat{\theta}$, *where* $\hat{\theta} = 1 - c$.

This limit property is intriguing and manifests a sharp discontinuity of the equilibrium set around $\sigma_x = 0$: a *tiny* perturbation away from perfect information suffices to obtain a unique equilibrium. Moreover, it has important implications for crises: as agents' private information becomes better and better, all non-fundamental volatility vanishes.

The key intuition behind these results is that disperse private information serves as an *anchor* for individual behavior: it limits the ability to forecast each others actions and thereby

---

monotone equilibrium is unique and the information is exogenous, a standard argument of iterrated deletion of strongly dominated strategies can be used to show that there are no other non-monotone equilibria either. Second, to prove the existence of multiple equilibria and the convergence to common-knowledge outcomes with either exogenous or endogenous information, it suffices to look at monotone equilibria.

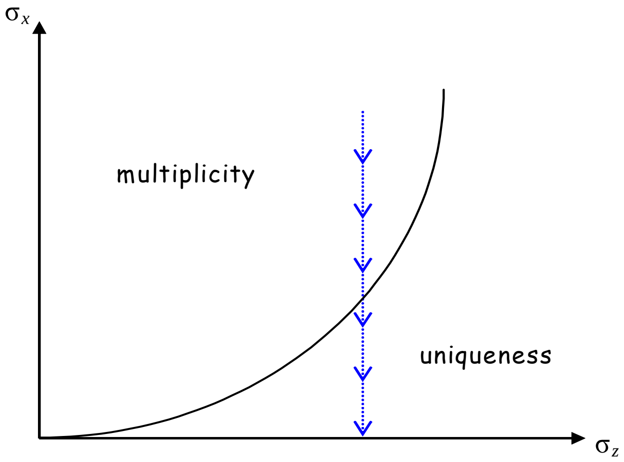

Figure 2: σ_x and σ_z parametrize the noise in private and public information; uniqueness is ensured for σ_x small enough.

coordinate on multiple equilibria. When all agents share the same information, they can perfectly forecast each others' actions in equilibrium. In particular, they can perfectly coordinate on either everyone or no one attacking. But when agents have heterogeneous information, each agent faces uncertainty regarding other agents' beliefs about θ and their actions, which makes coordination more difficult.

The higher the precision of private information, for given precision of public information, the more heavily individuals condition their actions on it, making it harder for other agents to predict their actions. When σ_x is sufficiently small relative to σ_z, this anchoring effect is strong enough that the ability to coordinate on multiple courses of action brakes down – a unique equilibrium then survives. Finally, as $\sigma_x \to 0$, individuals cease to condition their actions on the public signal and therefore the equilibrium dependence on the common noise ε vanishes.

3 Endogenous Information I: Financial Prices

The results reviewed above presume that the precision of public information remains invariant while varying the precision of private information. We now argue that this is unlikely to be the case when public information is endogenous, as it is with financial prices.

To investigate the information-aggregation role of prices, we introduce a financial market where agents trade a security prior to playing the coordination game. The dividend depends directly on the exogenous fundamentals or indirectly on the endogenous coordination game. In both cases, the equilibrium price conveys information valuable for the coordination game.

3.1 Model Setup

The model starts again with nature drawing θ from an improper uniform distribution over the real line and each agent receiving an exogenous private signal $x_i = \theta + \sigma_x \xi_i$. Agents then interact in two stages.

In stage 1, agents invest their wealth w in two assets. The first asset is riskless and is in infinitely elastic supply, with gross return normalized to 1. The second asset is risky, costs p in the first stage, and delivers a dividend $f = f(\theta, A)$ in the end of the second stage. The payoff agent i enjoys from his portfolio choice is $V(c_i) = -\exp(-\gamma c_i)/\gamma$, where $c_i = w - pk_i + fk_i$ denotes consumption, k_i investment in the risky asset, and $\gamma > 0$ the coefficient of absolute risk aversion.

The net supply of the risky asset is uncertain and not observed, given by $K^s(\varepsilon) = \sigma_\varepsilon \varepsilon$, where $\sigma_\varepsilon > 0$ and $\varepsilon \sim \mathcal{N}(0,1)$ and independent of θ and ξ_i. The role of the unobserved shock ε is to introduce noise in the information revealed by the market-clearing price; σ_ε thus parameterizes the exogenous noise in the aggregation process.

Stage 2 is essentially the same as the benchmark model of the previous section: agents chose whether to attack or not; the status quo is abandoned if and only if the mass of agents attacking exceeds θ; and the payoff of the agent from this stage is $U(a_i, A, \theta) = a_i (R(A, \theta) - c)$. The only difference is that agents now observe the price that cleared the financial market in stage 1. The regime outcome, the asset's dividend, and the payoffs from both stages are realized at the end of stage 2.[9]

In the absence of sunspots, individual asset demands and attack decisions are functions of x and p, the realizations of the private signal and the price. The corresponding aggregates are then functions of θ and p. We thus define an equilibrium as follows.

Definition 1 *An equilibrium is a price function, $P(\theta, \varepsilon)$, individual strategies for investment and attacking, $k(x,p)$ and $a(x,p)$, and their corresponding aggregates, $K(\theta,p)$ and $A(\theta,p)$, such that:*

$$k(x,p) \in \arg\max_{k \in \mathbb{R}} \mathbb{E}\left[\, V((f-p)\,k) \mid x, p \,\right] \tag{5}$$

$$K(\theta,p) = \int_x k(x,p)\phi\left(\tfrac{x-\theta}{\sigma_x}\right) dx \tag{6}$$

$$K(\theta, P(\theta,\varepsilon)) = K^s(\varepsilon) \tag{7}$$

---

[9] Note that preferences are separable between the first-stage portfolio choice and the second-stage attacking decision. Avoiding direct payoff linkages between the two stages allows us to isolate and focus on information aggregation.

$$a(x,p) \in \arg\max_{a\in[0,1]} \mathbb{E}[\; U(a, A(\theta,p), \theta) \mid x, p \;] \tag{8}$$

$$A(\theta,p) = \int_x a(x,p)\phi\left(\tfrac{x-\theta}{\sigma_x}\right)dx \tag{9}$$

Conditions (5)-(7) define a rational-expectations competitive equilibrium for stage 1: they require that individuals take into account all available information, including anything they can infer from the price realization $p = P(\theta, \varepsilon)$, and that the equilibrium price clears the asset market. Conditions (8)-(9) in turn define a Bayes-Nash equilibrium for stage 2: they require individuals to play best responses to each others, with beliefs pinned down by Bayes rule.[10]

Note that the equilibrium price is a function of the exogenous state (θ, ε). We similarly let $R^e(\theta,\varepsilon) = R(\theta, A(\theta, P(\theta,\varepsilon)))$ denote the equilibrium regime outcome.

3.2 Exogenous Dividend

We consider first the case that the dividend depends only on the exogenous fundamentals, namely $f = f(\theta)$. To preserve Normality, we let $f(\theta) = \theta$ and, following Grossman-Stiglitz, confine attention to equilibria with a linear price function. We also rule out perfectly revealing prices. The asset-market equilibrium we construct is unique within this space.[11]

With a linear price function, the observation of the price is equivalent to a Normal signal with precision some $\alpha_p \geq 0$, so that the posterior of θ conditional on x and p is Normal with mean $\delta x + (1-\delta)p$ and precision α, where $\delta = \alpha_x/\alpha$ and $\alpha = \alpha_x + \alpha_p$. Individual asset demands are thus given by

$$k(x,p) = \frac{\mathbb{E}[f|x,p] - p}{\gamma \mathrm{Var}[f|x,p]} = \frac{\delta\alpha}{\gamma}(x-p), \tag{10}$$

It follows that aggregate demand is $K(\theta,p) = (\delta\alpha/\gamma)(\theta - p)$ and therefore market clearing, $K(\theta,p) = K^s(\varepsilon) = \sigma_\varepsilon\varepsilon$, yields

$$P(\theta,\varepsilon) = \theta - \sigma_p\varepsilon, \tag{11}$$

where $\sigma_p = (\delta\alpha/\gamma)^{-1}\sigma_\varepsilon$. Thus, the assumption of a linear price function is confirmed, and $\alpha_p = \sigma_p^{-2}$. Combining with $\delta = \alpha_x/\alpha$ and $\alpha = \alpha_x + \alpha_p$, we obtain

$$\sigma_p = \gamma\sigma_\varepsilon\sigma_x^2, \tag{12}$$

which establishes that the precision of public information increases with the precision of private

---

[10] In general, one needs to allow the possibility of off-equilibrium realizations for p. For the equilibria we consider, however, there will be no such off-equilibrium events.

[11] As in Grossman-Stiglitz, a perfectly-revelaing equilibrium $p = \theta$ seems implausible; and it is not known whether there exist equilibria with non-linear price functions.

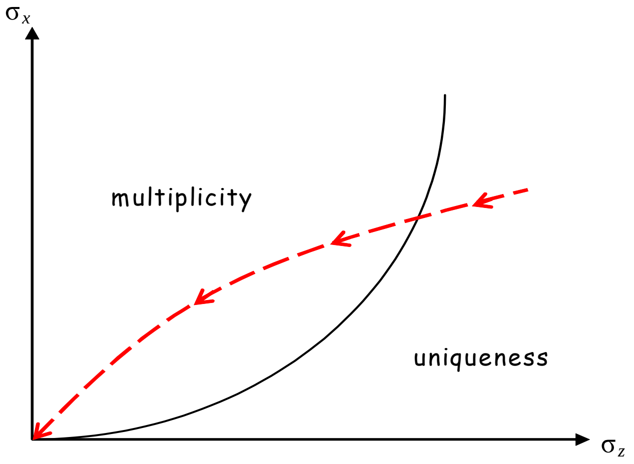

Figure 3: With endogenous public information, as σ_x decreases, σ_z also decreases; multiplicity is therefore ensured for sufficiently small σ_x.

information.

Turning to stage 2, we can apply the analysis of the benchmark model of Section 2 with the price, p, playing the role of the public signal, z. The agent attacks if and only if $x < x^*(p)$, the aggregate attack is $A(\theta, p) = \Phi\left(\sqrt{\alpha_x}\,(x^*(p) - \theta)\right)$, and the status quo is abandoned if and only if $\theta < \theta^*(p)$. The thresholds $x^*(p)$ and $\theta^*(p)$ are given by the solution to (2) and (4), replacing z with p (and α_z with α_p). Using (12), we conclude that the equilibrium at the coordination stage is unique if and only if $\sigma_x \leq \sigma_p^2\sqrt{2\pi}$, or equivalently $\sigma_\varepsilon^2\sigma_x^3 \geq \gamma^2(2\pi)^{-1/2}$.

Proposition 2 *In the asset economy with exogenous dividend $f = f(\theta)$, there are multiple equilibria (with a linear price function) if and only if $\sigma_\varepsilon^2\sigma_x^3 < \gamma^2(2\pi)^{-1/2}$. Multiplicity then emerges in the regime outcome, $R(\theta, \varepsilon)$, but not in the price function, $P(\theta, \varepsilon)$.*

In Proposition 1, σ_z were fixed and a sufficiently low σ_x ensured uniqueness. But here better private information implies also better public information, and at a rate fast enough to ensure multiplicity. The result is illustrated in Figure 3: in contrast to Figure 2: as σ_x decreases, σ_z also decreases, eventually entering the multiplicity region.

An immediate implication is that uniqueness can no more be seen as a small perturbation away from common knowledge: multiplicity is ensured when either σ_x or σ_ε are small, as illustrated in Figure 1. Indeed, the extreme common-knowledge outcomes can be recovered as either noise vanishes.

Corollary 2 *Consider the limit as $\sigma_x \to 0$ for given σ_ε, or the limit as $\sigma_\varepsilon \to 0$ for given σ_x. There exists an equilibrium in which $R(\theta, \varepsilon) \to 0$ whenever $\theta \in (\underline{\theta}, \overline{\theta})$, as well as an equilibrium in which $R(\theta, \varepsilon) \to 1$ whenever $\theta \in (\underline{\theta}, \overline{\theta})$. In every equilibrium, $P(\theta, \varepsilon) \to \theta$ for all (θ, ε).*

3.3 Endogenous Dividend

We now consider the case in which the payoff of the asset is endogenously determined by the coordination game, namely $f = f(A)$. We confine again attention to price systems that are not perfectly revealing and that preserve Normality of the information structure. For the same reason, we let $f(A) = -\Phi^{-1}(A)$.

Given a threshold $x^*(p)$ the aggregate attack is $A(\theta, p) = \Phi\left(\sqrt{\alpha_x}(x^*(p) - \theta)\right)$, so the dividend is

$$f = \sqrt{\alpha_x}(\theta - x^*(p)). \tag{13}$$

If θ were known, the equilibrium price would satisfy $p = f = \sqrt{\alpha_x}(\theta - x^*(p))$, or equivalently $\theta = p/\sqrt{\alpha_x} + x^*(p)$. With incomplete information, we hence guess (and later verify) that the posterior for θ is Normal with mean $\delta x + (1 - \delta)\tilde{p}$ and precision α, where $\delta = \alpha_x/\alpha$, $\alpha = \alpha_x + \alpha_p$ for some $\alpha_p \geq 0$, and

$$\tilde{p} = \frac{1}{\sqrt{\alpha_x}} p + x^*(p). \tag{14}$$

In this case, asset demands are given by

$$k(x, p) = \frac{\mathbb{E}[f|x, p] - p}{\gamma \text{Var}[f|x, p]} = \frac{\delta\alpha}{\tilde{\gamma}}(x - \tilde{p}),$$

and similarly $K(\theta, p) = (\delta\alpha/\widetilde{\gamma})(\theta - \widetilde{p})$, where $\widetilde{\gamma} = \gamma\sqrt{\alpha_x}$. Market clearing then pins down $\widetilde{p}$ as a unique linear combination of θ and ε :

$$\widetilde{p} = \theta - \sigma_p \varepsilon, \tag{15}$$

where $\sigma_p = (\delta\alpha/\tilde{\gamma})^{-1}\sigma_\varepsilon$.

To close the cycle, we consider equilibria with a one-to-one mapping between p and $\widetilde{p}$, in which case the observation of p is equivalent to the observation of $\tilde{p}$. Our guess regarding the posterior for θ is then verified with $\alpha_p = \sigma_p^{-2}$, or equivalently

$$\sigma_p = \gamma\sigma_\varepsilon\sigma_x^3. \tag{16}$$

That is, the precision of public information again increases with the precision of private information.

Turning to stage 2, the thresholds $\theta^*(p)$ and $x^*(p)$ solve the analogues of (2) and (3) from the benchmark model, once we replace z with $\tilde{p}$ and α_z with α_p:

$$x^*(p) = \theta^*(p) + \tfrac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*(p)), \tag{17}$$

$$\Phi\left(\sqrt{\alpha_x+\alpha_p}\big(\theta^*(p) - \tfrac{\alpha_x}{\alpha_x+\alpha_p}x^*(p) - \tfrac{\alpha_p}{\alpha_x+\alpha_p}\tilde{p}\big)\right) = c. \tag{18}$$

Substituting (14) and (17) into (18), we obtain a unique solution for $\theta^*(p)$:

$$\theta^*(p) = \Phi\left(\sqrt{\tfrac{\alpha_x}{\alpha_x+\alpha_p}}\Phi^{-1}(c) - \tfrac{\alpha_p}{\alpha_x+\alpha_p}p\right). \tag{19}$$

(17) then delivers also a unique $x^*(p)$. Unlike either the exogenous information benchmark or the case with an exogenous dividend, here the strategy of the agents in the coordination game and the regime outcome are *always* uniquely determined.

To complete the analysis, we determine the mapping between p and $\widetilde{p}$, which delivers the equilibrium price function $p = P(\theta,\varepsilon)$. Since $x^*(p)$ is uniquely determined, the aggregate demand, $K(\theta,p) = (\delta\alpha/\gamma)\left(\sqrt{\alpha_x}(\theta - x^*(p)) - p\right)$, is also uniquely determined. Moreover, $K(\theta,p)$ is continuous with respect to p, with $\lim_{p\to-\infty} K(\theta,p) = \infty$ and $\lim_{p\to+\infty} K(\theta,p) = -\infty$. It follows that the market-clearing condition $K(\theta,p) = K^s(\varepsilon)$, or equivalently equation (14), necessarily admits a solution. That is, an equilibrium price function, $p = P(\theta,\varepsilon)$, always exists.

We now show that the equilibrium price function (equivalently, the mapping between p and $\tilde{p}$) need not always be unique. Note that $\theta^*(p)$ and $x^*(p)$ are decreasing in p, which implies that the dividend $f = \sqrt{\alpha_x}(\theta - x^*(p))$ is increasing in the price p. As a result, the demand for the asset may be backward bending. Indeed, equations (17) and (**??**) imply

$$sign\left\{\frac{\partial K(\theta,p)}{\partial p}\right\} = -sign\left\{\frac{\sqrt{\alpha_x}}{\alpha_p} - \phi\left(\Phi^{-1}(\theta^*)\right)\right\}$$

so that $K(\theta,p)$ is globally decreasing in p if and only if $\sqrt{\alpha_x}/\alpha_p \geq \sqrt{2\pi}$, or equivalently $\sigma_\varepsilon^2\sigma_x^3 \geq \gamma^2/\sqrt{2\pi}$. When $\sigma_\varepsilon^2\sigma_x^3 < \gamma^2/\sqrt{2\pi}$, there is a non-empty interval $(\widetilde{p}_1, \widetilde{p}_2)$ such that (14) admits a single solution outside this interval and three solutions inside it.

Any selection from this correspondence that gives a one-to-one mapping from $\widetilde{p}$ to p sustains an equilibrium of the class we have considered. Different equilibrium price functions then generate different realizations of the price for the same exogenous state (θ,ε), and thus also different sizes of attack and regime outcomes.

Proposition 3 *In the asset economy with endogenous dividend $f = f(A)$, there are multiple equilibria (with a linear price function in terms of $\tilde{p}$) if and only if $\sigma_\varepsilon^2\sigma_x^5 < \gamma^2(2\pi)^{-1/2}$. Multiplicity then emerges in both the regime outcome, $R(\theta,\varepsilon)$, and the price function, $P(\theta,\varepsilon)$.*

When the dividend were exogenous, the price only played the role of a signal for the exogenous fundamental. As a result, multiplicity emerged solely in the coordination stage, not in the asset market. In contrast, now that the dividend depends on A, the price plays the role of an *anticipatory* signal of the size of the attack. In equilibrium, a higher price signals a higher dividend, implying that the demand for the asset may bend backwards over some region. This is the case when α_p is sufficiently high, for then the sensitivity of the agents' attack to the endogenous public signal and therefore of the dividend to the price is high. Non-monotonicity of the asset demand it turn explains the existence of multiple market-clearing prices.

Note that multiplicity emerges in the price and the regime outcome, but not in individual strategies: $x^*(p)$ was uniquely determined in (17), no matter the information structure. To gain some intuition on this result, it helps to consider the common-knowledge benchmark ($\sigma_\varepsilon = \sigma_x = 0$). In this case, $x = \theta$ and, in equilibrium, $p = f = -\Phi^{-1}(A)$. But then the agent finds it optimal to attack if and only if $A \geq \theta$, or equivalently $x \leq \Phi(-p)$. That is, the equilibrium strategy of the agent is uniquely determined as a function of x and p. However, the equilibrium p and A are not uniquely determined. Instead, for every $\theta \in (\underline{\theta}, \overline{\theta})$, both $(p, A) = (\infty, 0)$ and $(p, A) = (-\infty, 1)$ are consistent with an equilibrium.

These extreme common-knowledge equilibria are approached as either noise vanishes.

Corollary 3 *Consider the limit as $\sigma_x \to 0$ for given σ_ε, or the limit as $\sigma_\varepsilon \to 0$ for given σ_x. There is an equilibrium in which $R(\theta, \varepsilon) \to 1$ and $P(\theta, \varepsilon) \to -\infty$ whenever $\theta \in (\underline{\theta}, \overline{\theta})$, as well as an equilibrium in which $R(\theta, \varepsilon) \to 1$ and $P(\theta, \varepsilon) \to -\infty$ whenever $\theta \in (\underline{\theta}, \overline{\theta})$.*

3.4 Discussion

The key result of the analysis above is that the precision of endogenous public information increases with the precision of exogenous private information. This result is likely to be very robust.

An important implication of such a link between private and public information is that low values of private noise σ_x do not necessarily promote uniqueness of equilibria, as is the case in the exogenous-information setup. Instead, a race now ensues between private and public information.

In the examples considered above, public information wins the day: multiplicity is facilitated by a smaller σ_x because the precision of public information increases at a rate faster than the square root of the precision of private information. This, however, need not be true in general.

One possibility along these lines is that the dividend is not perfectly correlated with the fundamentals or the coordination outcome. We modify the previous section's examples and consider $f = \theta + \eta$ or $f = -\Phi^{-1}(A) + \eta$, where $\eta \sim \mathcal{N}\left(0, \sigma_\eta^2\right)$, independent of $(\theta, \xi, \varepsilon)$. Like the supply shock, η is another way to introduce noise in the aggregation process.

Proposition 4 *Suppose the dividend has an unknown random component* η. *For given* $\sigma_\eta > 0$ *and* $\sigma_\varepsilon > 0$

(i) *When* $f = \theta$, *a unique equilibrium survives* σ_x *sufficiently small.*

(ii) *When* $f = f(A) + \eta$, *multiple equilibria exist for* σ_x *sufficiently small.*

In either case, the region of $(\sigma_x, \sigma_\varepsilon)$ *for which there are multiple equilibria decreases with* σ_η.

The introduction of η reduces the informativeness of the equilibrium price, making coordination harder. A higher σ_η increases the riskiness of the asset, which reduces the sensitivity of demand to excess returns. Indeed, the sensitivity of demand to excess returns is now bounded from above by $1/\left(\gamma\sigma_\eta^2\right)$.

When the dividend is $f = \theta + \eta$ this implies that the sensitivity of demand to θ is also bounded from above. As a result, the precision of the price increases with the precision of private information, but remains bounded away from infinity. The direct effect of private information eventually dominates for σ_x small enough.

But when the dividend is $f = -\Phi^{-1}(A) + \eta$, the sensitivity of the aggregate attack and therefore of the dividend itself to θ increases with the precision of private information. This effect compensates and ensures that the precision of public information increases to infinity and fast enough so that multiplicity reemerges for small σ_x.

In Section 5, we consider a different model, where there is no financial market to aggregate information, but agents observe a direct signal about others' activity when they decide whether to attack. For this case, we find again that multiplicity holds when σ_x is small enough.

Hellwig, Mukherji, and Tsyvinski (2004), on the other hand, consider a currency-crises model in which the coordination game is embedded in the financial market: they assume that the dividend of the asset (i.e., a peso bond in their model) is itself low when the price of the asset is low (i.e., when interest rates are high). In their case, too, multiplicity is ensured for small levels of noise.

4 Noise and Volatility

We now investigate the role of the information structure for *non-fundamental* volatility, namely, the volatility of the regime outcome and the equilibrium price conditional on θ. We focus on the two CARA specifications examined above, with exogenous and endogenous dividend, and consider in turn two sources of non-fundamental volatility: volatility generated by payoff-irrelevant variables (sunspots) when there are multiple equilibria and agents use these sunspots to coordinate their behavior; and volatility generated by the non-fundamental shock ε when the equilibrium is unique.

With *exogenous* information, multiplicity disappears when agents observe the fundamentals with small idiosyncratic noise. By implication, there is no sunspot volatility when σ_x is small enough. What is more, as $\sigma_x \to 0$, the size of the attack and by implication the regime outcome become independent of ε. Hence, all non-fundamental volatility vanishes as $\sigma_x \to 0$.

With *endogenous* information, the impact of private noise on volatility is quite different. A sufficiently large reduction in σ_x can *increase* volatility by ensuring multiplicity and therefore making equilibrium outcomes depend on sunspots. Indeed, Corollaries 2 and 3 imply that sunspot volatility is maximized when either noise vanishes: as $\sigma_x \to 0$ or $\sigma_\varepsilon \to 0$, the regime can either collapse or survive for any given $\theta \in (\underline{\theta}, \overline{\theta}]$, purely as a function of the sunspot.[12] What is more, sunspot volatility can show up in prices as well: when the dividend is endogenous, as $\sigma_x \to 0$ or $\sigma_\varepsilon \to 0$, the equilibrium price can be arbitrarily low or arbitrarily high for any given $\theta \in (\underline{\theta}, \overline{\theta}]$.

The property that, with endogenous information, less noise may increase volatility does not rely on the existence of multiple equilibria. As we show next, when the equilibrium is unique, a reduction in either σ_x or σ_ε may increase the sensitivity of the regime outcome and the asset price to the exogenous shock ε and may therefore result to higher non-fundamental volatility.

4.1 Regime Volatility

In equilibrium, the regime is abandoned ($R = 1$) if and only if $\theta \leq \theta^* (p)$, but p in turn depends on θ, since $p = P(\theta, \varepsilon)$. To express the equilibrium regime outcome as a function of the exogenous variables θ and ε, note that, as long as the equilibrium is unique, $\theta^* (p)$ is continuously decreasing in p and $P(\theta, \varepsilon)$ is continuously increasing in θ. It follows that $R(\theta, \varepsilon) = 1$ if and only if $\theta \leq \hat{\theta}(\varepsilon)$, where $\hat{\theta}(\varepsilon)$ is the unique solution to $\hat{\theta} = \theta^*(P(\hat{\theta}, \varepsilon))$. We can thus examine the non-fundamental volatility of the regime outcome by examining the sensitivity of $\hat{\theta}(\varepsilon)$ to ε.

Consider the case the dividend is exogenous, $f = f(\theta)$. Using the results of Section 3.2, we get

$$\hat{\theta}(\varepsilon) = \Phi\left(\psi + \frac{1}{\gamma \sigma_\varepsilon \sigma_x} \varepsilon\right) \tag{20}$$

---

[12] Note that, unlike the case for σ_x, the impact of σ_ε on multiplicity and volatility is similar with exogenous and endogenous information.

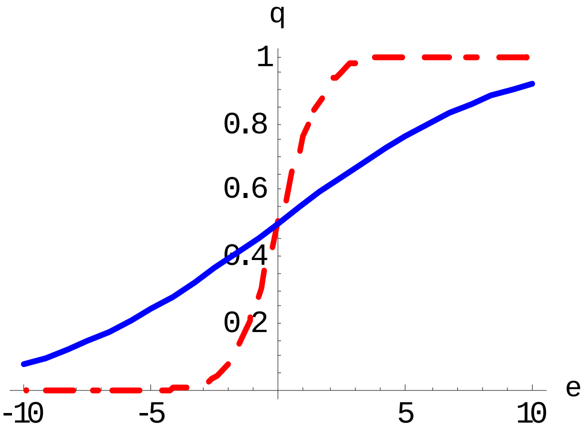

Figure 4: The regime-change threshold $\hat{\theta}(\varepsilon)$ as a function of the shock ε.

where $\psi = \sqrt{1 + 1/\left(\gamma^2 \sigma_\varepsilon^2 \sigma_x^4\right)} \Phi^{-1}(1-c)$. It follows that

$$\frac{\partial \hat{\theta}}{\partial \varepsilon} = \frac{\phi(\Phi^{-1}(\hat{\theta}))}{\gamma \sigma_\varepsilon \sigma_x} \tag{21}$$

and therefore, for any given $\hat{\theta}$, a reduction in either σ_ε or σ_x increases the slope of $\hat{\theta}(\varepsilon)$ with respect to ε. By implication, $\hat{\theta}$ satisfies a single-crossing property with respect to σ_x or σ_ε: let ε_0 be the unique value of ε for which $\partial\hat{\theta}/\partial\sigma_\varepsilon = 0$ or equivalently $\partial\hat{\theta}/\partial\sigma_x = 0$; for any ε_1 and ε_2 such that $\varepsilon_1 < \varepsilon_0 < \varepsilon_2$, we still have that $\partial|\hat{\theta}(\varepsilon_2) - \hat{\theta}(\varepsilon_1)|/\sigma_\varepsilon < 0$ and similarly $\partial|\hat{\theta}(\varepsilon_2) - \hat{\theta}(\varepsilon_1)|/\sigma_\varepsilon < 0$. In this sense, a reduction in either type of noise increases non-fundamental volatility.

The result is illustrated in Figure 4. The solid line depicts the threshold $\hat{\theta}(\varepsilon)$ as a function of ε for a relatively high σ_x (or σ_ε) whereas the dashed line corresponds to a relatively low σ_x (or σ_ε).

Proposition 5 *Less noise implies more non-fundamental volatility even when the equilibrium is unique: for any given $\hat{\theta}$, a reduction in σ_ε or σ_x increases the slope of $\hat{\theta}(\varepsilon)$ with respect to ε.*

Our earlier results regarding equilibrium multiplicity can thus be viewed as an extreme reincarnation of the above result. When the noise is sufficiently small, volatility can be high, not only because the outcome is very sensitive to the exogenous noise, but also because the outcome can depend on arbitrary sunspots.

When the dividend is endogenous, (20) and (21) continue to hold if we replace γ with $\tilde{\gamma} = \gamma/\sigma_x$. As a result, the sensitivity of $\hat{\theta}$ to ε increases with σ_ε but remains invariant with σ_x. Hence, a lower σ_x now can generate more non-fundamental volatility only by introducing multiplicity. A lower σ_ε, on the other hand, continues to generate more volatility also by increasing the sensitivity of the unique equilibrium. More generally, the sensitivity of $\hat{\theta}$ to ε depends on the relative precision of public information, which is likely to increase with a smaller σ_ε but not necessarily with a smaller σ_x.

4.2 Price Volatility

We next examine the comparative statics of the volatility of prices, focusing again on regions where the equilibrium is unique.

When the dividend is exogenous, we have $p = \theta - (\gamma \sigma_\varepsilon \sigma_x^2)\,\varepsilon$. The impact of noise on the sensitivity of the price to ε is then exactly like in Grossman-Stiglitz: a reduction in either σ_x or σ_ε implies lower non-fundamental volatility in prices.

When instead the dividend is endogenous, we have $p = f(A) - (\gamma \sigma_\varepsilon)\,\varepsilon$. Keeping the aggregate attack – and therefore the dividend – constant, the volatility of the price clearly decreases with a reduction in σ_ε and is independent of σ_x. But note that the dividend now is itself a function of ε, because agents' actions in the second stage depend on the price, which in turn depends on the shock ε. A reduction in σ_ε may thus have an ambiguous overall effect on price volatility by increasing the sensitivity of the coordination outcome, and therefore of the dividend, to the shock ε. We checked numerically that this indirect effect dominates in some cases. We thus conclude:

Proposition 6 *When the dividend is exogenous,* $f = \theta$, *the volatility of the price conditional on* θ *necessarily decreases with a reduction in either* σ_x *or* σ_ε. *But when the dividend depends on the outcome of the coordination game,* $f = f(A)$, *a reduction in* σ_ε *may increases price volatility.*

5 Endogenous Information II: Observable Actions

So far we have assumed that agents observe a signal from a financial market separated from the coordination game. We now remove the financial market and examine a case in which information originates in the coordination game itself: agents observe a public signal of the aggregate attack.

We first consider a model where the signal is about others *contemporaneous* actions. In this case, our equilibrium concept is novel and unavoidably at the crossroads of rational-expectations and game theory. We later show that the same results can be obtained in a dynamic variant of this model, in which the population is divided into two groups: 'early' movers, who have only private information about the fundamentals, and 'late' movers, who can also observe a public signal about the early movers' activity. In the latter case, the equilibrium concept is entirely game-theoretic.

5.1 Simultaneous signal

Model setup. There is no asset market and the coordination game builds on the benchmark model from Section 2. The new element is that all agents move simultaneously after observing private signals about the fundamentals *and* a public signal about the size of the attack. The private signals are as before $x_i = \theta + \sigma_x \xi_i$. The public signal is now

$$y = s(A, \varepsilon)$$

where $s : [0,1] \times \mathbb{R} \to \mathbb{R}$ and ε is noise, independent of θ and ξ. To preserve Normality of the information structure and obtain closed-form solution, we let $s(A, \varepsilon) = \Phi^{-1}(A) + \sigma_\varepsilon \varepsilon$ and $\varepsilon \sim \mathcal{N}(0,1)$.[13] The exogenous information structure is then parameterized by the pair of standard deviations $(\sigma_x, \sigma_\varepsilon)$.

Since the information of each agent includes a signal about other agents' actions, the equilibrium concept we define is a hybrid of a perfect Bayesian equilibrium and a rational-expectations equilibrium.

Definition 2 *An equilibrium consists of an endogenous signal $y = Y(\theta, \varepsilon)$, an individual attack strategy $a(x, y)$, and an aggregate attack $A(\theta, y)$, that satisfy:*

$$a(x,y) \in \arg\max_{a \in [0,1]} \mathbb{E}\left[\, U(a, A(\theta, y), \theta) \mid x, y \,\right] \tag{22}$$

$$A(\theta, y) = \int_x a(x,y) \phi\left(\tfrac{x-\theta}{\sigma_x}\right) dx \tag{23}$$

$$y = s(A(\theta, y), \varepsilon) \tag{24}$$

for all $(\theta, \varepsilon, x, y) \in \mathbb{R}^4$.

Condition (22) means that $a(x,y)$ is the optimal strategy for the agent, whereas condition (23) means that $A(\theta, y)$ is simply the aggregate across agents. The fixed-point relation introduced by (24) is the rational-expectations feature of our context: the signal y must be generated by individual actions, which in turn are contingent on y.

---

[13] This convenient specification was introduced by Dasgupta (2002).

Equilibrium analysis. We focus again on monotone equilibria, in which an agent attacks if and only if $x \leq x^*(y)$ and the status quo is abandoned if and only if $\theta \leq \theta^*(y)$. A monotone equilibrium is thus identified with a triplet of functions x^*, θ^*, and Y.

Given that an agent attacks if and only if $x \leq x^*(y)$, the aggregate attack is $A(\theta, y) = \Phi\left(\sqrt{\alpha_x}(x^*(y) - \theta)\right)$. It follows that the regime is abandoned if and only if $\theta \leq \theta^*(y)$, where $\theta^*(y)$ solves $A(\theta, y) = \theta$, or equivalently

$$x^*(y) = \theta^*(y) + \tfrac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*(y)). \tag{25}$$

Condition (24), on the other hand, implies that the signal must satisfy $y = \sqrt{\alpha_x}\left[x^*(y) - \theta\right] + \sigma_\varepsilon \varepsilon$, or equivalently

$$x^*(y) - \sigma_x y = \theta - \sigma_x \sigma_\varepsilon \varepsilon. \tag{26}$$

Note that (26) is a mapping between y and $z = \theta - \sigma_x \sigma_\varepsilon \varepsilon$. Define the correspondence

$$\mathcal{Y}(z) = \left\{ \; y \in \mathbb{R} \mid x^*(y) - \sigma_x y = z \; \right\}. \tag{27}$$

We will later show that $\mathcal{Y}(z)$ is non-empty and examine when it is single- or multi-valued.

Take any function $\tilde{Y}(z)$ that is a selection from this correspondence, $\tilde{Y}(z) \in \mathcal{Y}(z)$ for all z, and let $Y(\theta, \varepsilon) = \tilde{Y}(\theta - \sigma_x \sigma_\varepsilon \varepsilon)$. The observation of $y = Y(\theta, \varepsilon)$ is then equivalent to the observation of $z = \theta - \sigma_x \sigma_\varepsilon \varepsilon = Z(y)$, where $Z(y) \equiv x^*(y) - \sigma_x y$. Therefore, it is as if agents observed a Normal public signal about θ with noise

$$\sigma_z = \sigma_x \sigma_\varepsilon. \tag{28}$$

The precision of endogenous public information is increasing in the precision of exogenous private information, paralleling the results obtained with a financial price.

The individual attacks if and only if $x \leq x^*(y)$, where $x^*(y)$ solves the indifference condition

$$\Phi\left(\sqrt{\alpha_x + \alpha_z}\left(\theta^*(y) - \tfrac{\alpha_x}{\alpha_x + \alpha_z} x^*(y) - \tfrac{\alpha_z}{\alpha_x + \alpha_z} Z(y)\right)\right) = c. \tag{29}$$

Using $Z(y) = x^*(y) - \sigma_x y$ and substituting $x^*(y)$ from (25), we get

$$\theta^*(y) = \Phi\left(\frac{\alpha_z}{\alpha_x + \alpha_z} y + \sqrt{\frac{\alpha_x}{\alpha_x + \alpha_z}} \Phi^{-1}(1 - c)\right), \tag{30}$$

which together with (25) determines unique $\theta^*(y)$ and $x^*(y)$. The strategy $a(x, y)$ is thus uniquely determined.

We return to the equilibrium correspondence $\mathcal{Y}(z)$. Recall that this is given by the set of solutions to $x^*(y) - \sigma_x y = z$. Using (25) and (30) this reduces to

$$F(y) \equiv \Phi\left(\frac{\alpha_z}{\alpha_x + \alpha_z} y + \Lambda\right) + \frac{1}{\sqrt{\alpha_x}}\left(-\frac{\alpha_x}{\alpha_x + \alpha_z} y + \Lambda\right) = z, \tag{31}$$

where $\Lambda \equiv \sqrt{\alpha_x / (\alpha_x + \alpha_z)}\Phi^{-1}(1-c)$. Note that $F(y)$ is continuous in y, and $F(y) \to -\infty$ as $y \to +\infty$, and $F(y) \to +\infty$ as $y \to -\infty$, which ensures that $\mathcal{Y}(z)$ is non-empty, guaranteeing that an equilibrium exist. Next, note that

$$sign\left\{F'(y)\right\} = -sign\left\{1 - \frac{\alpha_z}{\sqrt{\alpha_x}}\phi\left(\tfrac{\alpha_z}{\alpha_x+\alpha_z} y + \Lambda\right)\right\}$$

and therefore $F(y)$ is globally monotonic if and only if $\alpha_z/\sqrt{\alpha_x} \leq \sqrt{2\pi}$, in which case $\mathcal{Y}(z)$ is single valued. If instead $\alpha_z/\sqrt{\alpha_x} > \sqrt{2\pi}$, there is a non-empty interval $(\underline{z}, \overline{z})$ within which $\mathcal{Y}(z)$ takes three values. Different selections then sustain different equilibria.

Using $\alpha_z = \alpha_\varepsilon \alpha_x$, we conclude that multiple equilibria survive as long as either source of noise is sufficiently small.[14]

Proposition 7 *A monotone rational-expectations equilibrium exists for all* $(\sigma_x, \sigma_\varepsilon)$ *and is unique if and only if* $\sigma_\varepsilon^2 \sigma_x \geq 1/\sqrt{2\pi}$. *If* $\sigma_\varepsilon^2 \sigma_x < 1/\sqrt{2\pi}$, *the equilibrium strategy* a *and aggregate attack* A *remain unique, but there are multiple signal functions* Y.

Finally, the common-knowledge outcomes are once again obtained as either source of information becomes infinitely precise. On the other hand, the Morris-Shin limit can be obtained if the endogenous public signal becomes infinitely imprecise.

Corollary 4 (*i*) *Consider the limit as either* $\sigma_x \to 0$ *or* $\sigma_\varepsilon \to 0$. *There exists an equilibrium in which* $R(\theta, \varepsilon) \to 0$ *whenever* $\theta \in (\underline{\theta}, \overline{\theta}]$, *as well as an equilibrium in which* $R(\theta, \varepsilon) \to 1$ *whenever* $\theta \in (\underline{\theta}, \overline{\theta}]$.

(*ii*) *Consider the limit as* $\sigma_\varepsilon \to \infty$. *There is a unique equilibrium in which* $R(\theta, \varepsilon) \to 1$ *whenever* $\theta < \hat{\theta}$ *and* $R(\theta, \varepsilon) \to 0$ *whenever* $\theta > \hat{\theta}$, *where* $\hat{\theta} \equiv 1 - c$.

5.2 Non-simultaneous signal

The analysis above has assumed that agents can condition their decision to attack on a noisy indicator of *contemporaneous* aggregate behavior. We now show that our results extend to a

---

[14] Interestingly, when multiplicity arises, it is with respect to aggregate outcomes, but not with respect to individual behavior. In this sense, the multiplicity here is similar to the one we encountered in Section 3.3, when agents traded a financial asset whose dividend depended on the size of the attack. Here, y is a direct signal about A; there, p was an indirect signal about A.

simple dynamic model in which no agent has information about contemporaneous actions of other agents and therefore a standard game-theoretic equilibrium concept (namely PBE) can be used.

Model setup. The population is divided into two groups, 'early' and 'late' agents. Early agents move first, on the basis of their private information alone. Late agents move second, on the basis of their private information as well as a noisy public signal about the aggregate activity of early agents. Neither group can observe contemporaneous activity, but late agents can condition their behavior on the activity of early agents.

Let $\mu \in (0,1)$ denote the fraction of early agents, A_1 the aggregate activity of early agents, and A_2 the aggregate activity of late agents. The signal generated by early agents and observed only by late agents is given by

$$y_1 = \Phi^{-1}(A_1) + \varepsilon, \tag{32}$$

where $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ is independent of θ and ξ. Early agents can condition their actions only on their private information, whereas late agents can condition their actions also on y_1. Finally, the regime changes if and only if $\mu A_1 + (1-\mu)A_2 \geq \theta$.

Equilibrium analysis. We look for perfect Bayesian equilibria in which the strategy of the agents in monotonic in their private information. Since late agents can condition their behavior on y_1 but early agents not, a monotone equilibrium is a scalar $x_1^* \in \mathbb{R}$ and a pair of functions $x_2^* : \mathbb{R} \to \mathbb{R}$ and $\theta^* : \mathbb{R} \to (0,1)$ such that: an early agent attacks if and only if $x \leq x_1^*$; a late agent attacks if and only if $x \leq x_2^*(y_1)$; and the regime is abandoned if and only if $\theta \leq \theta^*(y_1)$.

In such an equilibrium, the aggregate attack of early agents is $A_1(\theta) = \Phi\left(\sqrt{\alpha_x}\left[x_1^* - \theta\right]\right).$ It follows that $y_1 = \Phi^{-1}(A_1(\theta)) + \varepsilon = \sqrt{\alpha_x}\left[x_1^* - \theta\right] + \varepsilon$. Hence, in equilibrium, the observation of y_1 is equivalent to the observation of

$$z = x_1^* - \frac{1}{\sqrt{\alpha_x}} y_1 = \theta - \sigma_x \varepsilon,$$

which is a public signal with precision $\alpha_z = \alpha_\varepsilon \alpha_x$ (equivalently, with standard deviation $\sigma_z = \sigma_\varepsilon \sigma_x$), like in the simultaneous-signal model.

Since y_1 and z have the same informational content, we can equivalently express the strategy of a late agent as a function of z instead of y_1 and replace the functions $x_2^*(y_1)$ and $\theta^*(y_1)$ with $x_2^*(z)$ and $\theta^*(z)$. The aggregate attack of late agents is then $A_2(\theta, z) = \Phi\left(\sqrt{\alpha_x}\left[x_2^*(z) - \theta\right]\right)$ and the overall attack from both groups is $A(\theta, z) = \mu A_1(\theta) + (1-\mu) A_2(\theta, z)$. It follows that the regime changes if and only if $\theta \leq \theta^*(z)$, where $\theta^*(z)$ solves

$A\left(\theta^*(z), z\right) = \theta^*(z)$, or equivalently

$$\mu\Phi\left(\sqrt{\alpha_x}\left[x_1^* - \theta^*(z)\right]\right) + (1-\mu)\Phi\left(\sqrt{\alpha_x}\left[x_2^*(z) - \theta^*(z)\right]\right) = \theta^*(z). \tag{33}$$

Next, consider the optimal behavior of the agents. Since the realization of z is known to late agents, their decision problem is like in the benchmark model: the threshold $x_2^*(z)$ solves $\Pr\left[\theta \leq \theta^*(z) | x_2^*(z), z\right] = c$, or equivalently

$$\Phi\left(\sqrt{\alpha}\left(\delta x_2^*(z) + (1-\delta)\, z - \theta^*(z)\right)\right) = 1 - c, \tag{34}$$

where $\delta = \alpha_x/(\alpha_x + \alpha_z)$ and $\alpha = \alpha_x + \alpha_z$. Early agents, on the other hand, do not observe z and therefore face a double forecast problem: they are uncertain about both the fundamental and the signal upon which late agents will condition their behavior. The threshold x_1^* solves $\Pr\left[\theta \leq \theta^*(y) | x_1^*\right] = c$, or equivalently

$$\int \Phi\left(\sqrt{\alpha_x}\left[x_1^* - \theta^*(z)\right]\right) \sqrt{\alpha_1}\phi\left(\sqrt{\alpha_1}\left[x_1^* - z\right]\right) dz = 1 - c, \tag{35}$$

where $\alpha_1 = \alpha_x\alpha_\varepsilon/(1+\alpha_\varepsilon)$.[15]

A monotone equilibrium is therefore a joint solution to (33)-(35). We can reduce the dimensionality of the system by solving (33) for $x_2^*(z)$:

$$x_2^*(z) = \theta^*(z) + \frac{1}{\sqrt{\alpha_x}}\Phi^{-1}\left(\theta^*(z) + \frac{\mu}{1-\mu}\left\{\theta^*(z) - \Phi\left(\sqrt{\alpha_x}\left[x_1^* - \theta^*(z)\right]\right)\right\}\right).$$

Substituting the above into (34) and using $\delta = \alpha_x/(\alpha_x + \alpha_z)$ and $\alpha = \alpha_x + \alpha_z$, we obtain:

$$\Gamma(\theta^*(z), x_1^*) = g\left(z\right), \tag{36}$$

where

$$\Gamma(\theta, x_1) = -\frac{\alpha_z}{\sqrt{\alpha_x}}\theta + \Phi^{-1}\left(\theta + \frac{\mu}{1-\mu}\left\{\theta - \Phi\left(\sqrt{\alpha_x}\left[x_1^* - \theta\right]\right)\right\}\right),$$

$g\left(z\right) = \sqrt{1 + \alpha_z/\alpha_x}\Phi^{-1}\left(1-c\right) - \left(\alpha_z/\sqrt{\alpha_x}\right) z$, and $\alpha_z = \alpha_\varepsilon\alpha_x$. We conclude that an equilibrium is a joint solution of (35) and (36) for a threshold $x_1^* \in \mathbb{R}$ and a function $\theta^* : \mathbb{R} \to (0, 1)$.

Let $\mathcal{C}$ denote the set of piecewise continuous real functions with range a subset of $[0, 1]$. For any given function $\theta^* \in \mathcal{C}$, (35) always defines a unique $x_1^* \in \mathbb{R}$. For given $x_1^* \in \mathbb{R}$, on the other hand, (36) may admit a unique or multiple solutions in $\theta^* \in \mathcal{C}$, depending on $(\alpha_x, \alpha_\varepsilon, \mu)$. Different solutions to (36) for given x_1^* represent different continuation equilibria for the game

---

[15] To see this, note that $z = \theta - \sigma_x\varepsilon = x - \xi - \sigma_x\varepsilon$, so that $z|x \sim \mathcal{N}\left(0, \sigma_x^2 + \sigma_x^2\sigma_\varepsilon^2\right)$. That is, conditional on x, z is distributed normal with precision $\alpha_1 = \alpha_x\alpha_\varepsilon/(1+\alpha_\varepsilon)$.

between late agents defined by a fixed strategy for the early agents. The question of interest, however, is the determinacy of equilibrium in the entire game.

In the appendix we show that, when (36) admits a unique solution for *every* $x_1^* \in \mathbb{R}$, the fixed point of (35)-(36) is also unique. On the other hand, when (36) admits multiple solutions for *every* $x_1^* \in \mathbb{R}$, we prove that (35)-(36) also admits multiple fixed points. This provides us with the following sufficient (but not necessary) conditions for uniqueness and multiplicity:

Proposition 8 (**i**) *There exists a unique equilibrium if*

$$\sigma_\varepsilon^2 \sigma_x \geq \frac{1}{\sqrt{2\pi}} (1 - \mu)$$

(**ii**) *There exist multiple equilibria if*

$$\sigma_\varepsilon^2 \sigma_x < \frac{1}{\sqrt{2\pi}} \left(1 - \mu - \mu \sigma_\varepsilon^2\right)$$

For any σ_ε and σ_x such that $\sigma_\varepsilon^2 \sigma_x < 1/\sqrt{2\pi}$, multiplicity in ensured for μ low enough. In this sense, the multiplicity result of the simultaneous-move game survives as long as the fraction of informed (late) agents is high enough. Indeed, the dependence of (36) on x_1^* vanishes as $\mu \to 0$ and therefore it can be shown that the equilibria of the simultaneous-signal model can be approximated by equilibria of this dynamic model as $\mu \to 0$. What is more, for *any* $\mu < 1$, multiple equilibria exist as long as σ_ε and σ_x are sufficiently low. On the other hand, for any σ_x and any σ_ε, uniqueness is ensured by taking $\mu \to 1$, which is also intuitive, since in this case the role of the informed agents vanishes.

We conclude that the insights we derived in the simultaneous-signal model extend to the present framework and do not hinge on the fixed-point nature of the hybrid equilibrium concept we used there.

6 Final Remarks

We view the main theme in Morris-Shin as emphasizing the importance of the details of the information structure for understanding the determinacy of equilibria and the volatility of outcomes. This paper contributes to this same theme by studying the importance of endogenous information aggregation. We model public information by either (i) the price of a financial asset whose dividend depends on the underlying fundamentals or the outcome of the coordination game; or (ii) a direct noisy signal of others' activity in the coordination game. An important feature of the equilibrium in all cases is that the precision of endogenous public information rises with the precision of exogenous private information.

We showed that this effect is typically strong enough to reverse the limit uniqueness result obtained with exogenous information: multiplicity is now *ensured* when either the idiosyncratic noise in the individuals' observation of fundamentals or the common noise in the aggregation process is small enough; conversely, a unique equilibrium survives when the noise is large enough.

We also showed that less noise may have a destabilizing effect even when the equilibrium is unique: a reduction in either source of noise may increase the sensitivity of the coordination outcome and the price to exogenous shocks, thus leading to an increase in non-fundamental volatility. Our multiplicity result can thus be interpreted as an extreme version of this negative effect on volatility.

Our results on volatility may help understand crises phenomena such as currency attacks, bank runs, or debt crises. However, we have abstracted from the institutional details of each specific application, which may also be important for the questions of interest. A promising direction for future research is therefore to extend the analysis to particular applications, as well as to consider the welfare and policy implications.

7 Appendix

Proof of Proposition 1. Rewrite (4) as

$$G\left(\theta^*(z)\right) = g\left(z\right), \tag{37}$$

where $G(\theta) \equiv -\left(\alpha_z/\sqrt{\alpha_x}\right)\theta + \Phi^{-1}\left(\theta\right)$ and $g\left(z\right) = \sqrt{1+\alpha_z/\alpha_x}\Phi^{-1}\left(1-c\right) - \left(\alpha_z/\sqrt{\alpha_x}\right)z$. For every $z \in \mathbb{R}$, $G(\theta)$ is continuous in θ, with $G(0,z) = -\infty$ and $G(1,z) = \infty$, which implies that there necessarily exists a solution and any solution satisfies $\theta^*(z) \in (0,1)$. Next, note that

$$G'\left(\theta\right) = -\frac{\alpha_z}{\sqrt{\alpha_x}} + \frac{1}{\phi(\Phi^{-1}\left(\theta\right))}$$

and $\max_{w\in\mathbb{R}}\phi(w) = 1/\sqrt{2\pi}$. If $\alpha_z/\sqrt{\alpha_x} \leq \sqrt{2\pi}$ we have that G is strictly increasing in θ, which implies a unique solution to (37). If instead $\alpha_z/\sqrt{\alpha_x} > \sqrt{2\pi}$, then G is non-monotonic in θ and there is an interval $(\underline{z}, \overline{z})$ such that (37) admits multiple solutions $\theta^*(z)$ whenever $z \in (\underline{z}, \overline{z})$ and a unique solution otherwise. We conclude that monotone equilibrium is unique if and only if $\alpha_z/\sqrt{\alpha_x} \leq \sqrt{2\pi}$. **QED**

Proof of Corollary 1. Consider the limits as $\sigma_x \to 0$ for given σ_z, or $\sigma_z \to \infty$ for given σ_x. In either case, $\alpha_z/\sqrt{\alpha_x} \to 0$ and $\sqrt{(\alpha_x + \alpha_z)/\alpha_x} \to 1$. Condition (37) then implies that $\theta^*(z) \to \hat{\theta} = 1 - c$ for any z, so that the regime-change threshold is unique and independent of z. Similarly, $x^*(z) \to \hat{x}$, where $\hat{x} = \hat{\theta}$ if we consider the limit $\sigma_x \to 0$, and $\hat{x} = \hat{\theta} + \sigma_x \Phi^{-1}(\hat{\theta})$ if we instead consider the limit $\sigma_z \to \infty$. **QED.**

Proof of Propositions 3 and 4. See main text.

Proof of Proposition 4. *Part (i).* Postulating, like in Section 3.2, that the posterior for θ conditional on (x, p) is Normal with mean $\delta x + (1 - \delta)\, p$ and precision α, where $\delta = \alpha_x/\alpha$ and $\alpha = \alpha_x + \alpha_p$, we have the posterior for f is also Normal with the same mean and variance $\alpha^{-1} + \sigma_\eta$. Individual asset demands are thus given by

$$k(x,p) = \frac{\mathbb{E}[f|x,p] - p}{\gamma \text{Var}[f|x,p]} = \frac{\delta\,(x-p)}{\gamma\left(\alpha^{-1} + \sigma_\eta^2\right)}$$

and the equilibrium price by $p = \theta - \sigma_p \varepsilon$, where $\sigma_p = (\gamma/\delta)\left(\alpha^{-1} + \sigma_\eta^2\right)\sigma_\varepsilon = \gamma\left(\sigma_x^2 + \sigma_\eta^2/\delta\right)\sigma_\varepsilon$ Since $\delta \in [0, 1]$ and $\sigma_x > 0$, we have immediately that that σ_p is bounded from below by $\gamma\sigma_\eta^2\sigma_\varepsilon > 0$. It follows that $\sigma_x < \left(\gamma\sigma_\eta^2\sigma_\varepsilon\right)^2\sqrt{2\pi}$ suffices for $\sigma_x < \sigma_p^2\sqrt{2\pi}$ and hence for the equilibrium to be unique. Moreover, σ_p is increasing in σ_x and σ_ε, as well as σ_η. A higher σ_η thus makes it more likely that the equilibrium is unique.

Part (ii). Like in Section 3.2, let the posterior for θ be Normal with mean $\delta x + (1 - \delta)\,\tilde{p}$ and precision α, with $\delta = \alpha_x/\alpha$, $\alpha = \alpha_x + \alpha_p$, and $\tilde{p}$ as in (14). It follows that

$$k(x,p) = \frac{\mathbb{E}[f|x,p] - p}{\gamma \text{Var}[f|x,p]} = \frac{\sqrt{\alpha_x}\delta\,(x - \tilde{p})}{\gamma\left(\alpha_x\alpha^{-1} + \sigma_\eta^2\right)} = \frac{\delta}{\tilde{\gamma}\left(\alpha^{-1} + \sigma_\eta^2\alpha_x^{-1}\right)}\,(x - \tilde{p})\,,$$

where $\tilde{\gamma} = \gamma\sqrt{\alpha_x}$, and therefore $\tilde{p} = \theta - \sigma_p\varepsilon$, where $\sigma_p = \tilde{\gamma}\left(\alpha^{-1} + \sigma_\eta^2\alpha_x^{-1}\right)\sigma_\varepsilon/\delta$. Using $\delta\alpha = \sigma_x^{-2}$, $\tilde{\gamma} = \gamma/\sigma_x$ and $1/\delta = 1 + \sigma_p/\sigma_x$, we get

$$\sigma_p = \tilde{\gamma}\left(1/\left(\delta\alpha\right) + \sigma_\eta^2\sigma_x^2/\delta\right)\sigma_\varepsilon = \tilde{\gamma}\sigma_x^2\left(1 + \sigma_\eta^2/\delta\right)\sigma_\varepsilon = \gamma\sigma_x\left(1 + \sigma_\eta^2\left(1 + \sigma_p/\sigma_x\right)\right)\sigma_\varepsilon$$

and therefore

$$\sigma_p = \frac{1 + \sigma_\eta^2}{1 - \gamma\sigma_\eta^2\sigma_\varepsilon}\gamma\sigma_\varepsilon\sigma_x.$$

We conclude that a higher σ_η again makes it harder for multiple equilibria to exist, nevertheless multiplicity in ensured by a sufficiently small σ_x or σ_ε.

Proof of Propositions 5, 6, and 7. See main text.

Proof of Corollary 4. *Part (i).* From conditions (29) and (30) we have that, for every y, $\theta^*(y) \to 1 - c = \hat{\theta}$ and $x^*(y) \to \hat{\theta} + \sigma_x \Phi^{-1}(\hat{\theta}) = \hat{x}$ as $\sigma_\varepsilon \to \infty$. Condition (26) then implies $\theta - \sigma_x \varepsilon = x^*(y) - \sigma_x y \to \hat{x} - \sigma_x y$ and therefore the unique signal function in the limit is $Y(\theta, \varepsilon) \to (\hat{x} - \theta)/\sigma_x + \varepsilon$.

Part (ii). First, note that $\underline{y} \to -\infty$ and $\overline{y} \to +\infty$ as $\sigma_x \to 0$. Next, note that both $|\sigma_\varepsilon^2 \sigma_x - \phi(\underline{y})|$ and $|\sigma_\varepsilon^2 \sigma_x - \phi(\overline{y})|$ vanish. Since $\lim_{y \to -\infty} \phi(y) y = \lim_{y \to +\infty} \phi(y) y = 0$, the latter implies $\sigma_x \underline{y} \to 0$ and $\sigma_x \overline{y} \to 0$. Hence, $\underline{z} \to \Phi(-\infty) = \underline{\theta}$ and $\overline{z} \to \Phi(+\infty) = \overline{\theta}$ as $\sigma_x \to 0$. Moreover, for every θ and ε, $\theta - \sigma_x \varepsilon \to \theta$ as $\sigma_x \to 0$. It follows that

$$\Pr\left[\ \theta - \sigma_x \varepsilon \in (\underline{z}, \overline{z}) \mid \theta \in (\underline{\theta}, \overline{\theta})\ \right] \to 1 \quad \text{as} \quad \sigma_x \to 0.$$

Next, let $\underline{Y}(\theta, \varepsilon) \equiv \min \mathcal{Y}(\theta - \sigma_x \varepsilon)$ and $\overline{Y}(\theta, \varepsilon) \equiv \max \mathcal{Y}(\theta - \sigma_x \varepsilon)$ and consider (θ, ε) such that $\theta - \sigma_x \varepsilon \in (\underline{z}, \overline{z})$. Note that $\underline{Y}(\theta, \varepsilon) < \underline{y} < \overline{y} < \overline{Y}(\theta, \varepsilon)$ and therefore

$$\underline{Y}(\theta, \varepsilon) \to -\infty \quad \text{and} \quad \overline{Y}(\theta, \varepsilon) \to +\infty \quad \text{as} \quad \sigma_x \to 0.$$

From (30), $\theta^*(y)$ is independent of σ_x, $\theta^*(y) \to \Phi(-\infty) = \underline{\theta}$ as $y \to -\infty$, and $\theta^*(y) \to \Phi(+\infty) = \overline{\theta}$ as $y \to +\infty$. It follows that, as long as $\theta \in (\underline{\theta}, \overline{\theta})$,

$$\Pr\left[\ \theta \leq \theta^*\left(\underline{Y}(\theta, \varepsilon)\right)\ \right] \to 0 \quad \text{and} \quad \Pr\left[\ \theta \leq \theta^*\left(\overline{Y}(\theta, \varepsilon)\right)\ \right] \to 1 \quad \text{as} \quad \sigma_x \to 0,$$

which establishes the result. **QED**

Proof of Proposition 8. For any $\mu \in (0,1)$ and any $x_1^* \in \mathbb{R}$, $\Gamma(\theta, x_1^*)$ is continuous in θ, with $\Gamma(\underline{\theta}, x_1^*, \mu) = -\infty$ and $\Gamma(\overline{\theta}, x_1^*, \mu) = \infty$, where $\underline{\theta} = \underline{\theta}(x_1^*, \alpha_x, \mu)$ and $\overline{\theta} = \overline{\theta}(x_1^*, \alpha_x, \mu)$ solves, respectively, $\theta + \frac{\mu}{1-\mu}\left\{\theta - \Phi\left(\sqrt{\alpha_x}\left[x_1^* - \theta\right]\right)\right\} = 0$ and $= 1$, and therefore satisfy $0 < \underline{\theta} < \overline{\theta} < 1$. It follows that (36) always admits a solution $\theta^*(z) \in \left(\underline{\theta}, \overline{\theta}\right)$. That is, for any given $x_1^* \in \mathbb{R}$, (36) defines at least one function $\theta^* : \mathbb{R} \to \left(\underline{\theta}, \overline{\theta}\right)$.

We next examine under what conditions the function that solves (36) is unique. Note that

$$\frac{\partial \Gamma}{\partial \theta} = -\frac{\alpha_z}{\sqrt{\alpha_x}} + \Lambda\left(\theta; x_1^*, \alpha_x, \mu\right)$$

where

$$\Lambda\left(\theta; x_1^*, \alpha_x, \mu\right) \equiv \frac{1}{\phi\left(\Phi^{-1}\left(\theta + \frac{\mu}{1-\mu}\left\{\theta - \Phi\left(\sqrt{\alpha_x}\left[x_1^* - \theta\right]\right)\right\}\right)\right)} \left\{1 + \frac{\mu}{1-\mu}\left[1 + \sqrt{\alpha_x}\phi\left(\sqrt{\alpha_x}\left[x_1^* - \theta\right]\right)\right]\right\}$$

As $\theta \to \underline{\theta}$ or $\overline{\theta}$ (equivalently, $z \to \pm\infty$), $\Lambda\left(\theta; x_1^*, \alpha_x, \mu\right) \to +\infty$. Let

$$K\left(x_1^*, \alpha_x, \mu\right) \equiv \inf_{\theta \in \mathbb{R}} \Lambda\left(\theta; x_1^*, \alpha_x, \mu\right)$$

and note that, since ϕ takes values in $(0, 1/\sqrt{2\pi}]$,

$$K\left(x_1^*, \alpha_x, \mu\right) \geq \frac{1}{1/\sqrt{2\pi}}\left\{1 + \frac{\mu}{1-\mu}\left[1 + \sqrt{\alpha_x} 0\right]\right\} = \frac{\sqrt{2\pi}}{1-\mu}.$$

Moreover, letting $\hat{\theta} = \hat{\theta}\left(x_1^*, \alpha_x, \mu\right) \in \left(\underline{\theta}, \overline{\theta}\right)$ be the solution to

$$\phi\left(\Phi^{-1}\left(\theta + \frac{\mu}{1-\mu}\left\{\theta - \Phi\left(\sqrt{\alpha_x}\left[x_1^* - \theta\right]\right)\right\}\right)\right) = 1/\sqrt{2\pi},$$

or equivalently the solution to $\theta + \frac{\mu}{1-\mu}\left\{\theta - \Phi\left(\sqrt{\alpha_x}\left[x_1^* - \theta\right]\right)\right\} = 1/2$, and using again the fact that the maximal value of $\phi(.)$ is $1/\sqrt{2\pi}$, we have

$$\begin{aligned} K\left(x_1^*, \alpha_x, \mu\right) &\leq \Lambda\left(\hat{\theta}; x_1^*, \alpha_x, \mu\right) = \frac{1}{1/\sqrt{2\pi}}\left\{1 + \frac{\mu}{1-\mu}\left[1 + \sqrt{\alpha_x}\phi\left(\sqrt{\alpha_x}\left[x_1^* - \hat{\theta}\right]\right)\right]\right\} \\ &\leq \sqrt{2\pi}\left\{1 + \frac{\mu}{1-\mu}\left[1 + \frac{\sqrt{\alpha_x}}{\sqrt{2\pi}}\right]\right\} \end{aligned}$$

Combining, we conclude that, for all $\left(x_1^*, \alpha_x, \mu\right)$,

$$\overline{K}\left(\alpha_x, \mu\right) \geq K\left(x_1^*, \alpha_x, \mu\right) \geq \underline{K}\left(\mu\right)$$

where

$$\overline{K}\left(\alpha_x, \mu\right) \equiv \sqrt{2\pi}\left\{1 + \frac{\mu}{1-\mu}\left[1 + \frac{\sqrt{\alpha_x}}{\sqrt{2\pi}}\right]\right\} \quad \text{and} \quad \underline{K}\left(\mu\right) \equiv \frac{\sqrt{2\pi}}{1-\mu}.$$

Note that, importantly, neither bound is a function of x_1^*.

Case (i) : $\sigma_\varepsilon^2 \sigma_x \geq \frac{1}{\sqrt{2\pi}}(1-\mu)$.

In this case, $\frac{\alpha_z}{\sqrt{\alpha_x}} \leq \underline{K}(\mu) \leq \inf_{x_1^*} K\left(x_1^*, \alpha_x, \mu\right)$ and therefore Γ is strictly increasing in θ for all θ and all x_1^*. It follows that (36) defines a unique function $\theta^* : \mathbb{R} \to \left(\underline{\theta}, \overline{\theta}\right)$ for any given x_1^*. Moreover, since Γ is decreasing in x_1^* and g is decreasing in z, the function θ^* is decreasing in z and increasing in x_1^*. Finally, θ^* is continuous in both z and x_1^*.

Next, consider (35). For any given $\theta^* : \mathbb{R} \to \left(\underline{\theta}, \overline{\theta}\right)$, (35) admits a unique solution $x_1^* \in \mathbb{R}$. Moreover, this solution is continuous and increasing in θ^*.

Let C be the set of continuous (and bounded) functions $\theta^* : \mathbb{R} \to \left(\underline{\theta}, \overline{\theta}\right)$. Then, (36) is a mapping $\mathbb{R} \to C$ and (35) is a mapping $C \to \mathbb{R}$. Together, they define a continuous and increasing mapping $T : \mathbb{R} \to \mathbb{R}$.

It is easy to check that $T(-\infty) > -\infty$ and $T(+\infty) < \infty$. Hence, a fixed point always exists. Moreover, for arbitrary x_1^* and $a > 0$, let $x_1^{**} = x_1^* + a$ and let θ^* and θ^{**} be the solutions to (36) for x_1^* and x_1^{**}, respectively, that is,

$$\begin{aligned}
\Gamma\left(\theta^*, x_1^*\right) &= -\frac{\alpha_z}{\sqrt{\alpha_x}}\theta^* + \Phi^{-1}\left(\frac{1}{1-\mu}\theta^* - \Phi\left(\sqrt{\alpha_x}\left[x_1^* - \theta^*\right]\right)\right) = g \\
\Gamma\left(\theta^{**}, x_1^{**}\right) &= -\frac{\alpha_z}{\sqrt{\alpha_x}}\theta^{**} + \Phi^{-1}\left(\frac{1}{1-\mu}\theta^{**} - \Phi\left(\sqrt{\alpha_x}\left[x_1^* + a - \theta^{**}\right]\right)\right) = g
\end{aligned}$$

Then (35) gives

$$\begin{aligned}
\int \Phi\left(\sqrt{\alpha_x}\left[Tx_1^* - \theta^*(z)\right]\right)\sqrt{\alpha_1}\phi\left(\sqrt{\alpha_1}\left[Tx_1^* - z\right]\right)dz &= 1-c \\
\int \Phi\left(\sqrt{\alpha_x}\left[Tx_1^{**} - \theta^{**}(z)\right]\right)\sqrt{\alpha_1}\phi\left(\sqrt{\alpha_1}\left[Tx_1^{**} - z\right]\right)dz &= 1-c
\end{aligned}$$

Since $\frac{\alpha_z}{\sqrt{\alpha_x}} < \frac{\sqrt{2\pi}}{1-\mu}$, we clearly have $\theta^{**} < \tilde{\theta} = \theta^* + a$ and, by implication, $Tx_1^{**} < \tilde{x}_1$, where $\tilde{x}_1$ solves

$$\int \Phi\left(\sqrt{\alpha_x}\left[\tilde{x}_1 - \theta^*(z) - a\right]\right)\sqrt{\alpha_1}\phi\left(\sqrt{\alpha_1}\left[\tilde{x}_1 - z\right]\right)dz = 1-c$$

If $\tilde{x}_1 = Tx_1^* + a$ $(> Tx_1^*)$, the above would have been positive, so it must be that $\tilde{x}_1 < Tx_1^* + a$. Therefore, $Tx_1^{**} < Tx_1^* + a$, which proves that the slope of the mapping T is less than one for every x_1^*. It follows that T has a unique fixed point.

Case (ii) : $\sigma_\varepsilon^2\sigma_x < \frac{1}{\sqrt{2\pi}}\left(1 - \mu - \mu\sigma_\varepsilon^2\right)$.

In this case, $\frac{\alpha_z}{\sqrt{\alpha_x}} > \overline{K}\left(\alpha_x, \mu\right) \geq \sup_{x_1^*} K\left(x_1^*, \alpha_x, \mu\right)$ and therefore Γ necessarily has a non-empty region of non-monotonicity in θ for all x_1^*. It follows that, for any x_1^*, there is a non-empty interval $Z = (\underline{z}, \overline{z}) = Z\left(x_1^*\right)$ such that (36) admits three distinct solutions whenever $z \in Z$ and a unique one otherwise. Let θ_L^* (resp., θ_H^*) be the function defined by selecting the lowest (resp., highest) solution whenever $z \in Z$ and the unique one whenever $z \notin Z$. Let T_L (resp., T_H) be the associated mappings. Each of the mappings T_L and T_H are continuous and satisfy $T(-\infty) > -\infty$ and $T(+\infty) < \infty$. Hence, there exists a fixed point (at least one) for each mapping. Moreover, for any given x_1^*, $\theta_L^*(z) < \theta_H^*(z)$ for all $z \in Z$, which implies (because of the monotonicity in (35) and the fact that Z has positive measure) that $T_L\left(x_1^*\right) < T_H\left(x_1^*\right)$ for any x_1^*. It follows that the lowest fixed point of T_L is lower than the highest fixed point of T_H, which together with the fact that $\theta_L^* < \theta_H^*$ for any given x_1^* implies that the associated x_2^* and θ^* satisfy the same ordering.

References

[1] Angeletos, George-Marios, Christian Hellwig, and Alessandro Pavan (2003) "Coordination and Policy Traps," MIT/UCLA/Northwestern working paper.

[2] Angeletos, George-Marios, Christian Hellwig, and Alessandro Pavan (2004) "Information Dynamics and Equilibrium Multiplicity in Global Games of Regime Change," MIT/UCLA/Northwestern working paper.

[3] Angeletos, George-Marios, and Alessandro Pavan (2004) "Transparency of Information and Coordination in Economies with Investment Complementarities," *American Economic Review* 94 (*P&P*).

[4] Atkeson, Andrew (2000), "Discussion on Morris and Shin," *NBER Macro Annual.*

[5] Barlevy, Gadi, and Pietro Veronesi (2003), "Rational Panics and Stock Market Crashes," *Journal of Economic Theory* 110, 234-263.

[6] Calvo, G. (1988), "Servicing the Public Debt: the Role of Expectations." *American Economic Review,* 78(4), 647-661.

[7] Carlsson, Hans, and Eric van Damme (1993), "Global Games and Equilibrium Selection," *Econometrica* 61, 5, 989-1018.

[8] Chari, V.V., and Patrick Kehoe (2003), "Hot Money," *Journal of Political Economy* 111, 1262-1292.

[9] Cole, Harold, and Timothy Kehoe (1996), "Self-fulfilling Debt Crises," Federal Reserve Bank of Minneapolis staff report 211.

[10] Cooper, Russell (1998), *Coordination Games, Complementarities and Macroeconomics*, Cambridge University Press.

[11] Cooper, Russell, and Andrew John (1988), "Coordinating Coordination Failures in Keynesian Models," *Quarterly Journal of Economics* 103, 441–463.

[12] Corsetti, Giancarlo, Bernardo Guimaraes, and Nouriel Roubini, (2004), "International Lending of Last Resort and Moral Hazard: A Model of IMF's Catalytic Finance," working paper, University of Rome.

[13] Dasgupta, Amil (2002), "Coordination, Learning and Delay," working paper, London School of Economics.

[14] Diamond, Douglas, and Philip Dybvig (1983), "Bank Runs, Deposit Insurance, and Liquidity," *Journal of Political Economy* 91, 401-19.

[15] Goldstein, Itay, and Ady Pauzner (2000), "Demand Deposit Contracts and the Probability of bank Runs," working paper, Duke University and Tel Aviv University.

[16] Grossman, Sanford (1981) "An Introduction to the Theory of Rational Expectations under Asymmetric Information," Review of Economic Studies, 48, 541-559.

[17] Grossman, Sanford, and Joseph Stiglitz (1976), "Information and Competitive Price Systems," *American Economic Review* 66, 246-253.

[18] Hellwig, Martin (1980) "On the Aggregation of Information in Competitive Markets" *Journal of Economic Theory*, 22, 477-498.

[19] Hellwig, Christian (2002), "Public Information, Private Information, and the Multiplicity of Equilibria in Coordination Games," *Journal of Economic Theory*, 107, 191-222.

[20] Hellwig, Christian, Arijit Mukherji, and Alej Tsyvinski (2004), "Coordination Failures and Asset Prices," working paper, UCLA.

[21] Morris, Stephen, and Hyun Song Shin (1998), "Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks", *American Economic Review*, 88, 3, 587-597.

[22] Morris, Stephen, and Hyun Song Shin (1999), "Private versus Public Information in Coordination Problems", Yale/Oxford mimeo.

[23] Morris, Stephen, and Hyun Song Shin (2000), "Rethinking Multiple Equilibria in Macroeconomics," *NBER Macroeconomics Annual 2000*, 139-161.

[24] Morris, Stephen, and Hyun Song Shin (2002), "The Social Value of Public Information", *American Economic Review* 92, 1521-1534.

[25] Morris, Stephen, and Hyun Song Shin (2003), "Global Games - Theory and Applications," in M. Dewatripont, L. Hansen, and S. Turnovsky, eds., *Advances in Economics and Econometrics*, Cambridge, UK: Cambridge University Press.

[26] Morris, Stephen, and Hyun Song Shin (2004a), "Coordination Risk and the Price of Debt," *European Economic Review* 48, 133-153.

[27] Morris, Stephen, and Hyun Song Shin (2004b), "Liquidity Black Holes," *Review of Finance* 8, 1-18.

[28] Obstfeld, Maurice (1986), “Rational and Self-Fulfilling Balance-of-Payments Crises”, *American Economic Review* 76, 1, 72-81.

[29] Obstfeld, Maurice (1996), “Models of Currency Crises with Self-Fulfilling Features,” *European Economic Review* 40, 3-5, 1037-47.

[30] Rochet, Jean-Charles, and Xavier Vives (2004), “Coordination Failures and the Lender of Last Resort: Was Bagehot Right After All?,” working paper, University of Toulouse.

[31] Tarashev, Nikola (2003), “Currency Crises and the Informational Role of Interest Rates,” Bank of International Settlements Discussion Paper.

[32] Velasco, Andres (1996), “Fixed Exchange Rates: Credibility, Flexibility and Multiplicity,” *European Economic Review* 40, 3-5, 1023-36.