General Functional Forms for Age-Efficiency Functions

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The literature on capital deterioration has highlighted the straightline, one-hoss shay, and geometric forms, which are pedagogically useful and have simple age-price profiles. Tests of which form best describes actual age-related efficiency losses are hampered, however, for want of an overall form that subsumes all three as limiting cases and has a convenient age-price function. The leading research in the field in fact begins with the flexible Box-Cox age-*price* function [Hulten and Wykoff, 1981a and 1981b; Koumanakos and Hwang, 1988; Hulten, Wykoff, and Robertson, 1989], from which the corresponding efficiency form can be derived, but that approach forces the discount rate into the efficiency function. While it is not critical to keep the discount rate out of the efficiency function, one would like its presence or absence to be the modeler's choice. But it is more important to guarantee, without resorting to undue parameter restrictions or changes in numbering conventions, that the age-efficiency profile never increase with age. As will be shown, the Box-Cox method can have problems here, at least in the single-asset case where compositional effects do not boil down an entire cohort of assets to something roughly geometric.

By contrast, this paper works up from the efficiency side, from which the slope restriction carries over naturally to a well-behaved age-price function. The idea is to "trap" simple functions that, once domesticated, are surprisingly flexible. Moreover, the trapped expressions solve linear differential equations relating efficiency loss to surviving efficiency, and so may be construed as approximations to any downward-sloping age-efficiency function. Of the expressions to be presented, only one gives rise to an easy age-price dual—i.e., something expressible with the functions available in most econometric software, without recourse to infinite series—but even that one can distinguish among the three pedagogical forms. Higher-order age-price integrals require higher functions or infinite-series expressions, so regressions using them would be within reach only of analytical math software such as *Mathematica*. But the higher-order integrals in turn can be approximated by conditional sums of the easy first-order form. The paper emphasizes the pure effects of age on the efficiency and fair resale price of a single asset. Time-*cum*-vintage effects are secondary, although the resulting age-based algebraic forms suggest a natural normalization. And while most econometric treatments so far have estimated *cohort* depreciation rates, it will be important, for the day when resale prices and the distribution of lifespans are modeled *jointly*, to have flexible specifications that work for individual assets.

Constructive Approach: "The Trap"

Of the many age-efficiency functions that have been proposed, the Geometric is best known and easiest to use:

$$\Phi(s) = e^{\delta s}, \tag{1}$$

where s represents the age of the asset and $\delta<0$ is both the rate of *deterioration* of service flows and the rate of *depreciation* of the resale price—a self-dual property shared by no other form. An asset that decays geometrically loses the largest fraction of its original efficiency, as well as of its original value, when it is brand new, but its service flows and resale price never quite reach zero.[1] These properties may seem implausible for single assets, but most empirical studies[2] show the assumption is not far wrong for the *cohorts* of assets from which perpetual-inventory capital aggregates are tabulated. Still, in the individual case, one might want to cut off remaining service-flows for ages above some to-be-estimated positive lifespan, L:

$$e^{\delta s} - e^{\delta L},$$

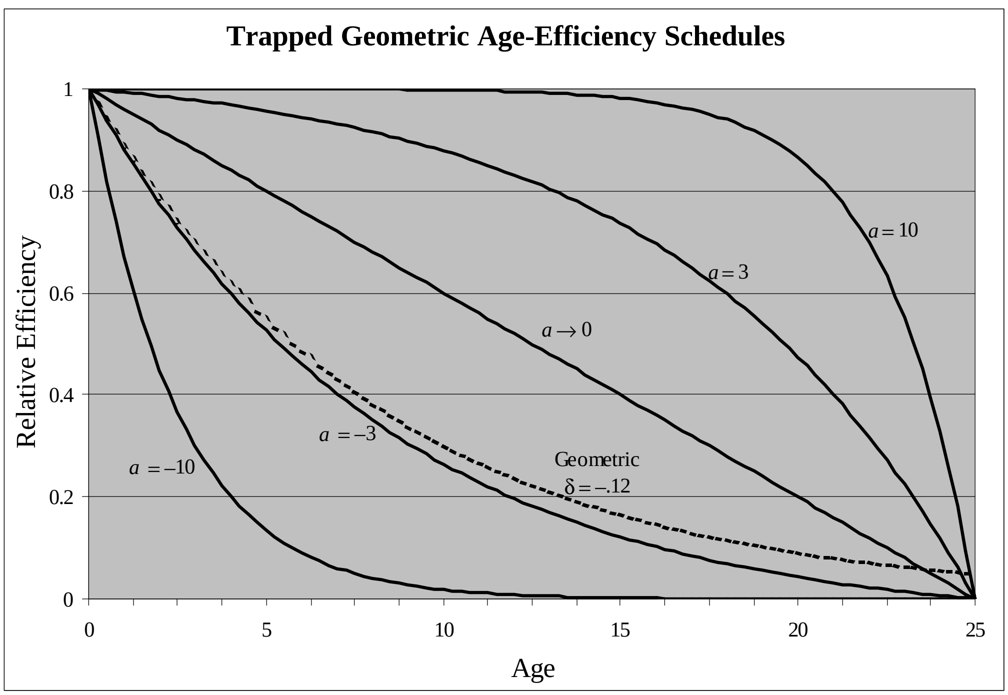
and then normalize the difference by the corresponding gap when the asset is new:

$$(e^{\delta s} - e^{\delta L})/(e^{\delta 0} - e^{\delta L}).$$

Replacing δ by a/L to avoid a units problem yields the "Trapped Geometric" form:

$$\Phi(s/L) = \frac{e^{as/L} - e^{a}}{1 - e^{a}} \,. \qquad (2)$$

Unlike the ordinary Geometric δ, the parameter a need not be negative. Positive values of a in fact yield shapes that are concave downward, ranging from the straightline (as $a \to 0$) to the one-hoss shay (as $a \to \infty$). Further, for negative a, replacing a by δL and allowing $L \to \infty$ returns the original Geometric. The family of Trapped Geometric schedules fills the age-efficiency rectangle with "one-hump" curves that never cross. Several Trapped Geometric assets with 25-year lifespans, as well as a comparable Geometric asset, are drawn here:



[1] Plainly the resale price of an old Geometric asset would eventually fall below scrap value, and the resulting retirement and dismemberment would likely terminate its remaining service flows.
[2] See Fraumeni [1997] for an extensive survey.

The trap works on other functions as well. Consider the Harmonic age-efficiency function:

$$\Phi(s) = 1/(1-\eta s), \tag{3}$$

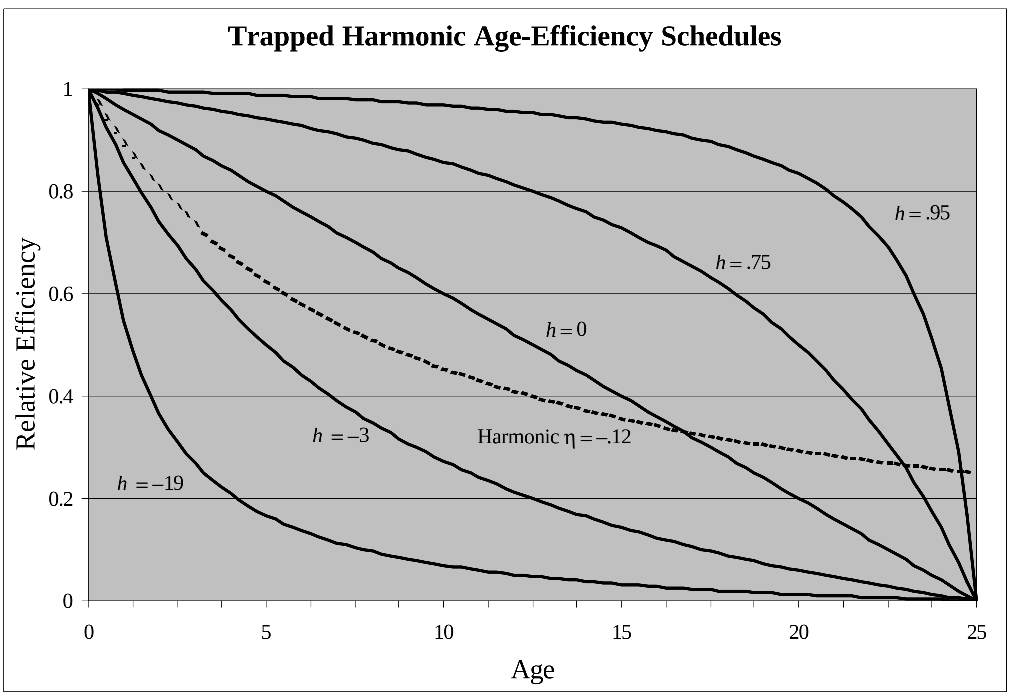
which resembles the Geometric for $\eta<1$. The trapped form:

$$\frac{\dfrac{1}{1-hs}-\dfrac{1}{1-hL}}{\dfrac{1}{1-h0}-\dfrac{1}{1-hL}}$$

reduces to:

$$\Phi(s/L) = \frac{L-s}{L-hs}\,, \tag{4}$$

upon replacing η by h/L. So the "Trapped Harmonic" function is just the Hyperbolic form introduced by Jack Faucett Associates [1973] and subsequently used by the Bureau of Labor Statistics [1983], which allows concave shapes for positive values of h between the straightline (h=0) and one-hoss shay (h=1) cases, as well as convex shapes for any negative h. Replacing negative h by ηL and taking $L\to\infty$ returns the original Harmonic. Like Trapped Geometric curves, the family of Trapped Harmonic / Hyperbolic schedules fills the age-efficiency rectangle with non-intersecting "one-hump" curves:



but the Hyperbolic curves treat the vertical and horizontal axes symmetrically (hence "hyperbolic"). In other words, a schedule (not drawn) connecting the points of greatest curvature of a family of Trapped Harmonic profiles would be a simple straight segment from the origin to the top-right corner of the rectangle; the corresponding schedule for the Trapped Geometric family would have a lazy-S shape.

A final example is more involved. Consider the Logistic age-efficiency function:

$$\Phi(\mathrm{s}) = \frac{e^{d\mathrm{s}}}{1 - m(1 - e^{d\mathrm{s}})}, \tag{5}$$

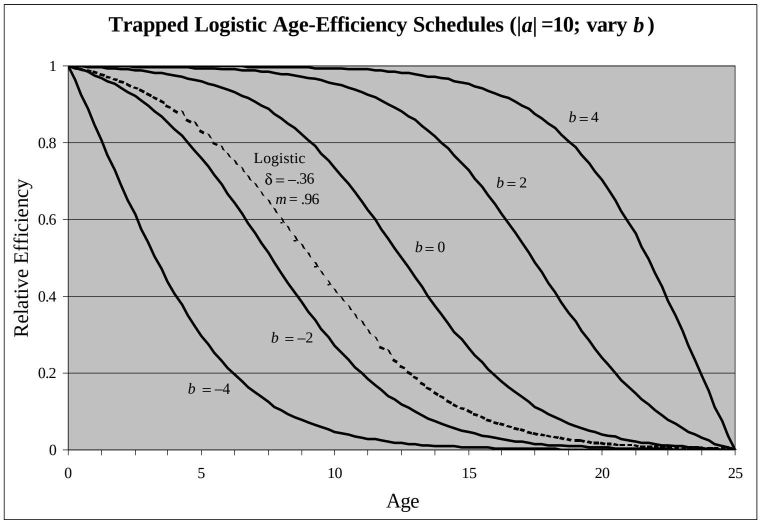
which begins at 1 when s=0 and tends toward 0 as $s\rightarrow\infty$, provided $\delta<0$ and $m<1$. The curve resembles the Geometric for large s, but may take on a "backward-S" appearance at younger ages for m between ½ and 1. If m is in that range, the inflection point occurs at $(s^*, \Phi(s^*)) = \left(\frac{ln(\frac{1}{m} - 1)}{d}, \frac{1}{2m}\right)$—which is to say the inflection points of the family of downward-sloping Logistic functions occur at $\Phi(s^*) = ½(1+e^{\delta s^*})$—so no further inflection would occur in the decay pattern of a Logistic asset that is at least half "used up." Applying the trap mechanically gives:

$$\frac{e^{d\mathrm{s}} - e^{d\mathrm{L}}}{(1 - e^{d\mathrm{L}})[1 - m(1 - e^{d\mathrm{s}})]},$$

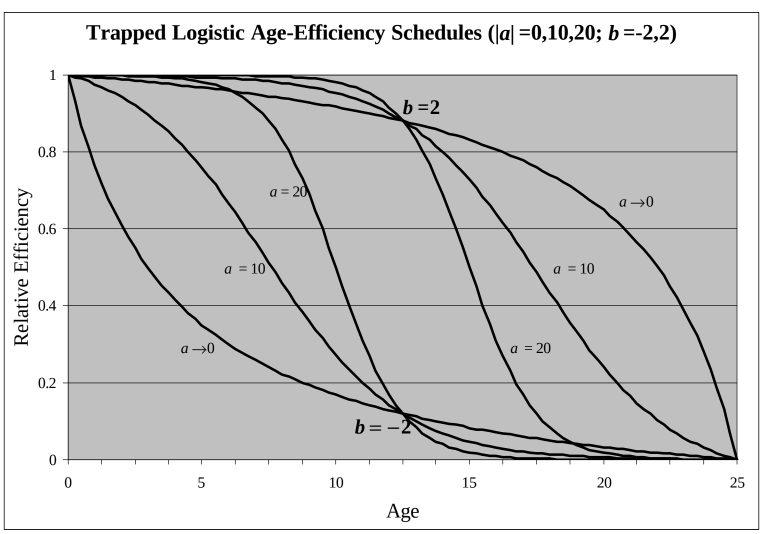
but the resulting expression is inconvenient. The parameter δ (or better yet: $a = \delta\mathrm{L}$) now may take either sign, as in the Trapped Geometric case, but the restriction on m is harder to implement: $m(1-e^{a})<1$. Also, the same curve may be represented by the parameters $(a, m) = (a_o, m_o)$, or by $(a, m) = (-a_o, 1-m_o)$. So replace m by $\frac{e^{-b} - e^{-a/2}}{e^{a/2} - e^{-a/2}}$ to find the "Trapped Logistic" form:

$$\Phi(\mathrm{s/L}) = \frac{e^{a\mathrm{s/L}} - e^{a}}{e^{a\mathrm{s/L}} - e^{a} + e^{a/2-b}\left(1 - e^{a\mathrm{s/L}}\right)} = \frac{Sinh\left[\frac{a}{2}\left(1 - \frac{\mathrm{s}}{\mathrm{L}}\right)\right]}{Sinh\left[\frac{a}{2}\left(1 - \frac{\mathrm{s}}{\mathrm{L}}\right)\right] + e^{-b} Sinh\left[\frac{a}{2}\frac{\mathrm{s}}{\mathrm{L}}\right]}, \tag{6}$$

which is indifferent to the sign of a and allows b to assume any real value. The expression specializes to the Trapped Geometric when $b=\pm a/2$ and to the Trapped Harmonic as $a\rightarrow 0$ (so $h=1-e^{-b}$). Replacing a by $\delta\mathrm{L}$ (given $\delta<0$) and $e^{a/2-b}$ by $1-m(1-e^{\delta\mathrm{L}})$, then taking the limit as $\mathrm{L}\rightarrow\infty$, returns the original Logistic. The form can fill the age-efficiency rectangle with "one-hump" curves that may intersect, but also with a wider variety of "backward-S" shapes than the unconstrained Logistic. If b is between $-|a|/2$ and $|a|/2$, inflection points are real and occur at $(s^*, \Phi(s^*/\mathrm{L})) = \left(\frac{\mathrm{L}}{a} ln\left(\frac{e^{a/2} - e^{-b}}{e^{-b} - e^{-a/2}}\right), \frac{e^{b} - \left(e^{a/2} + e^{-a/2}\right)/2}{\left(e^{b} + e^{-b}\right) - \left(e^{a/2} + e^{-a/2}\right)}\right)$ —which is to say the inflection points of the family of downward-sloping Trapped Logistic functions occur at $\Phi(s^*/\mathrm{L}) = \frac{1}{2}\left(1 + \frac{Sinh[a(\frac{1}{2} - \frac{s^*}{\mathrm{L}})]}{Sinh[\frac{a}{2}]}\right)$—so no inflection would occur outside the two "eighths" of the age-efficiency rectangle bounded by Φ=½ and Φ=1–s/L. In practical terms, b=0 implies an inflection point exactly in the center of the age-efficiency rectangle: $(s^*, \Phi(s^*/\mathrm{L}))$ = (½L, ½); negative b shifts the inflection earlier and higher, until $b=ln[2e^{a/2}/(1+e^{a})]$ implies $(s^*, \Phi(s^*/\mathrm{L}))$ = (0, 1); positive b shifts it later and lower, until $b = ln[½(1+e^a)/e^{a/2}]$ implies $(s^*, \Phi(s^*/L)) = (L, 0)$. Curves with the same *b* but different *a* always intersect in the interior of the rectangle at $(s, \Phi) = (½L, (1+e^{-b})^{-1})$. Larger absolute values of *a* make inflected schedules curve more sharply; as *a* grows very large, *b* defines a "waterfall" form that plunges from nearly 1 to barely 0 when $s^* \approx (½ + b/|a|)L$. A couple graphs will help. First consider:



which collects a family of backward-S curves having inflection points inside the 25-year lifespan. The curves are not very S-shaped at all when *b*= –4 or 4: given *a*=10, it turns out that *b*= –4.307 places the inflection point at (s,Φ)=(0,1) while *b*=4.307 puts the inflection point at (s,Φ)=(L,0). Setting *b*= –5 would force the inflection infinitely far into the past, and *b*=5 would push it remotely far ahead; these same values of *b* reduce the schedule to the Trapped Geometric (with *a*=–10 and 10, respectively). In other words, two "opposite" Trapped Geometric curves (i.e., parametrized by *a* and –*a*) define a "lens" within which Trapped Logistic curves of the same |*a*| have real inflection points. Next consider:



which shows two clusters of curves, now organized about *b* rather than *a*. Note the two curves with a=20 are sharply inflected and essentially parallel, with inflection points at roughly 40 and 60 percent of the lifespan: this is the "waterfall" effect for large *a*. Note also that the two curves with $a \to 0$ are actually disguised Trapped Harmonic / Hyperbolic functions, with $h \approx -6.39$ for $b = -2$ and $h \approx .86$ for b=2.

The upshot of all this is that trapped age-efficiency schedules are remarkably flexible. The Trapped Geometric function covers a continuum of forms including all three pedagogical shapes. The Trapped Harmonic does nearly as well (but misses the Geometric), and the Trapped Logistic contains both as special cases plus backward-S shapes. In addition, all three explicitly specify the lifespan. The next section presents a somewhat more rigorous characterization of this flexibility.

Deductive Approach: Linear Differential Equations as Approximations to Arbitrary Efficiency Functions

Suppose a researcher is fortunate enough to observe the marginal products arising from a single asset as it ages, but not the precise formula of the asset's age-efficiency function. To estimate the unknown formula, the researcher might fit the observations to the *real* parameters of a first-order linear differential equation:

$$\Phi'(\mathrm{s}) = A + B\,\Phi(\mathrm{s}),$$

or for greater flexibility, to the real parameters of a first-order quadratic differential equation:

$$\Phi'(\mathrm{s}) = A + B\,\Phi(\mathrm{s}) + C\,\Phi(\mathrm{s})^2,$$

abandoning higher-power effects to the unstated error term. But such regressions risk simultaneity bias, for Φ (and perhaps its square) appears both as a regressor and, in differential form, as the dependent variable. Although powers of age are available as instruments, the differential equations do have closed-form solutions; finding them would show how and where to use age in small-sample, albeit nonlinear, forms. So solve the differential equations, subject to the slope condition $\Phi'(s) \le 0$ from s=0 through s=L and the boundary conditions Φ(0)=1 and Φ(L)=0.

Solving the linear equation is easy. The integral:

$$\int \frac{d\Phi}{A + B\Phi} = \int d\mathrm{s},$$

has the general solution $\ln[A + B\,\Phi]/B = \mathrm{s} + k$, with k the arbitrary constant. Impose the new-asset condition Φ(0)=1 to find $k = \ln[A + B]/B$. Apply k and impose the expiration condition Φ(L)=0 to find $A = Be^{BL}/(1-e^{BL})$. Finally, apply both k and A to the general solution to find:

$$\Phi = \frac{e^{B\mathrm{s}} - e^{B\mathrm{L}}}{1 - e^{B\mathrm{L}}},$$

which slopes downward, given positive L, for any real *B*. This is the Trapped Geometric form, with δ=B.

Solving the quadratic equation is harder. First simplify the integral:

$$\int \frac{d\Phi}{A + B\Phi + C\Phi^2} = \int d\mathrm{s}$$

to:

$$\frac{2}{\sqrt{4AC - B^2}} \int \frac{du}{1 + u^2} = \int d\mathrm{s}$$

by the change of variables: $u = (B + 2C\Phi)\big/\sqrt{4AC - B^2}$. The general solution, expressed in terms of Φ, is $\frac{2}{\sqrt{4AC-B^2}} ArcTan[\frac{B+2C\Phi}{\sqrt{4AC-B^2}}] = s + k$, where again k is the arbitrary constant. The sign of $4AC–B^2$ matters. For now, let $\delta^2 = B^2–4AC > 0$, and use the Logarithmic representation, $ArcTan[u] = \frac{i}{2} ln[\frac{1-iu}{1+iu}]$, to reduce the general solution to: $ln[\frac{d-B-2C\Phi}{d+B+2C\Phi}]\big/d = s + k$. Next impose the new-asset condition Φ(0)=1 to find $k = ln[\frac{d-B-2C}{d+B+2C}]\big/d$, then apply k and the terminal condition Φ(L)=0 to find $C = \frac{(d^2-B^2)(e^{d\mathrm{L}}-1)}{2(d-B+(d+B)e^{d\mathrm{L}})}$. Finally, apply both k and C to the general solution to find:

$$\Phi = \frac{e^{d\mathrm{s}} - e^{d\mathrm{L}}}{1 - e^{d\mathrm{L}}} \cdot \frac{d\left(1 + e^{d\mathrm{L}}\right) - B\left(1 - e^{d\mathrm{L}}\right)}{d\left(e^{d\mathrm{s}} + e^{d\mathrm{L}}\right) - B\left(e^{d\mathrm{s}} - e^{d\mathrm{L}}\right)},$$

which slopes downward, given positive L and any real δ, provided $B \geq d\,\frac{1+e^{d\mathrm{L}}}{1-e^{d\mathrm{L}}}$. Since this is hard to implement, replace B by $d\left(\frac{1+e^{d\mathrm{L}}}{1-e^{d\mathrm{L}}} - 2\,\frac{e^{d\mathrm{L}/2+b}}{1-e^{d\mathrm{L}}}\right)$, which is useful but not obvious,[3] and simplify until:

$$\Phi = \frac{e^{d\mathrm{s}} - e^{d\mathrm{L}}}{e^{d\mathrm{s}} - e^{d\mathrm{L}} + e^{d\mathrm{L}/2-b}\left(1 - e^{d\mathrm{s}}\right)},$$

which is the Trapped Logistic form.

In other words, to track an arbitrary decay process parsimoniously *to a linear approximation,* use the Trapped Geometric form. Settling for the linear approximation confines the Φ function to "one-hump" curves. To track the same process *to a quadratic approximation*, subject to the regularity condition $B^2–4AC>0$, use the Trapped Logistic form, which allows both one-hump curves and "two-hump" backward-S curves.

It is worth examining the regularity condition further. Suppose it fails; that is, suppose $\delta^2 = 4AC–B^2 > 0$, so that i does not cancel from the arctangent. Now follow the steps that led to the Trapped Logistic solution,

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[3] B and b are one-to-one. Note in particular: $\lim_{b\to\infty} B \to \infty$, $\lim_{b\to\pm\delta\mathrm{L}/2} B = \pm\delta$, $\lim_{b\to 0} B = \delta(1–e^{\delta\mathrm{L}/2})/(1+e^{\delta\mathrm{L}/2})$, and $\lim_{b\to-\infty} B = \delta(1+e^{\delta\mathrm{L}})/(1–e^{\delta\mathrm{L}})<0$.

but use $i\delta$ instead of δ and replace B by $i\,d\left(\frac{1+e^{id\mathrm{L}}}{1-e^{id\mathrm{L}}} - 2\,\frac{e^{id\mathrm{L}/2+b}}{1-e^{id\mathrm{L}}}\right) = \delta\left(e^{b}Csc[\delta\mathrm{L}/2] - Cot[\delta\mathrm{L}/2]\right)$.[4] The result resembles the Trapped Logistic algebraically, but behaves in the opposite manner, allowing two-hump "reclining-chair" shapes instead of backward-S profiles. Writing this "Irregular Trapped Logistic" form as:

$$\Phi(\mathrm{s/L}) = \frac{e^{ias/\mathrm{L}} - e^{ia}}{e^{ias/\mathrm{L}} - e^{ia} + e^{ia/2-b}\left(1 - e^{ias/\mathrm{L}}\right)} = \frac{Sin\left[\frac{a}{2}\left(1-\frac{\mathrm{s}}{\mathrm{L}}\right)\right]}{Sin\left[\frac{a}{2}\left(1-\frac{\mathrm{s}}{\mathrm{L}}\right)\right] + e^{-b}Sin\left[\frac{a}{2}\frac{\mathrm{s}}{\mathrm{L}}\right]}, \qquad (7a)$$

with $a = \delta\mathrm{L}$, shows the close connection to the regular form. However, while b may assume any real value, the cyclicality of the sine function necessitates that a lie between -2π and 2π for continuity (again the *sign* of a has no effect). It is more convenient to replace a by $4ArcTan[z]$, where z has the sign of a but covers the whole real line:

$$\Phi(\mathrm{s/L}) = \frac{Sin\left[2\left(1-\frac{\mathrm{s}}{\mathrm{L}}\right)ArcTan[z]\right]}{Sin\left[2\left(1-\frac{\mathrm{s}}{\mathrm{L}}\right)ArcTan[z]\right] + e^{-b}Sin\left[2\,\frac{\mathrm{s}}{\mathrm{L}}\,ArcTan[z]\right]}, \qquad (7z)$$

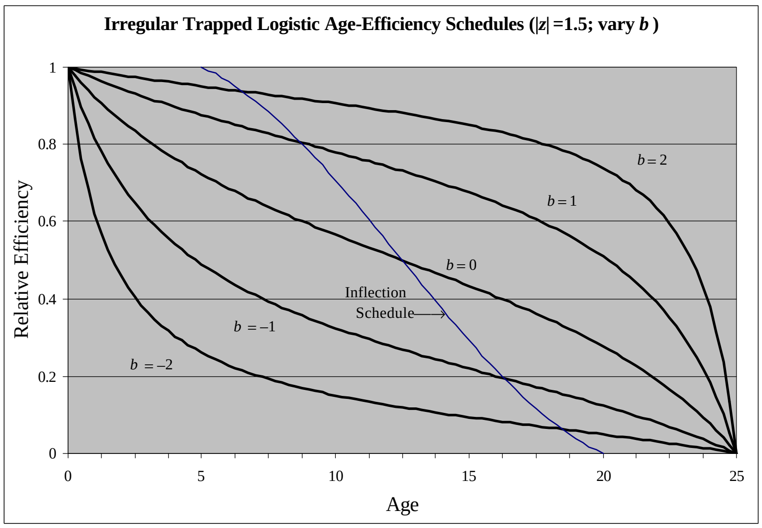
The expression reduces to the Trapped Harmonic as $z \to 0$ ($h = 1 - e^{-b}$ again), but has no other obvious specializations. A real inflection point occurs when $Cos\left[2\,\frac{s^*}{\mathrm{L}}\,ArcTan[z]\right] = e^{b}Cos\left[2\left(1-\frac{s^*}{\mathrm{L}}\right)ArcTan[z]\right]$. This does not restrict real b, unlike the regular Trapped Logistic case, but it does impinge on s^*/L. The coordinates of the inflection point may be written as parametric functions of a (or z) and b: $\left(s^*, \Phi(s^*/\mathrm{L})\right)$ =

$$\left(\frac{\mathrm{L}}{i\,a}ln\left(\frac{e^{ia/2} - e^{-b}}{e^{-b} - e^{-ia/2}}\right), \frac{e^{b} - \left(e^{ia/2} + e^{-ia/2}\right)/2}{\left(e^{b} + e^{-b}\right) - \left(e^{ia/2} + e^{-ia/2}\right)}\right) = \left(-\frac{\mathrm{L}}{2}\frac{ArcTan\left[\frac{1-z^2}{2z} - e^{-b}\,\frac{1+z^2}{2z}\right]}{ArcTan[z]}, \frac{e^{b} - \frac{1-z^2}{1+z^2}}{e^{b} + e^{-b} - 2\,\frac{1-z^2}{1+z^2}}\right)$$

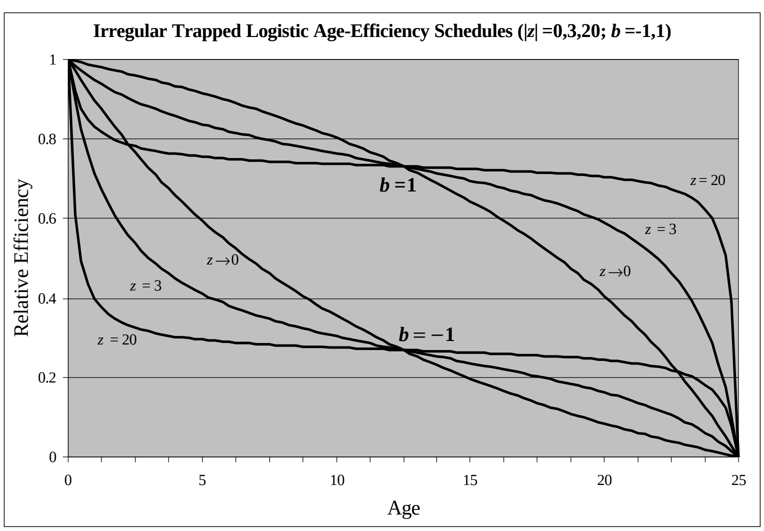
—which is to say that *allowed* inflection points of downward-sloping Irregular Trapped Logistic functions occur when $\Phi(s^*/\mathrm{L}) = \frac{1}{2}\left(1 + \frac{(1+z^2)Sin\left[2(1-2\,\frac{s^*}{\mathrm{L}})ArcTan[z]\right]}{2z}\right)$ is between 0 and 1. Real inflection age s^* must therefore fall between $\frac{p}{4|ArcTan[z]|}\mathrm{L}$ and $\left(1 - \frac{p}{4|ArcTan[z]|}\right)\mathrm{L}$, so for $0 \le s^* \le \mathrm{L}$, no inflection would occur outside the two "eighths" of the age-efficiency rectangle bounded by s=½L and Φ=1–s/L. In practical terms, b=0 puts an inflection point at the center of the age-efficiency rectangle: $\left(s^*, \Phi(s^*/\mathrm{L})\right)$ = (½L, ½). Negative b shifts the inflection point lower and (depending on z) later: for $-1 \le z \le 1$, reducing b from zero to $ln\left[\frac{1-z^2}{1+z^2}\right]$ implies $\left(s^*, \Phi(s^*/\mathrm{L})\right)$ = (L, 0); for $|z| > 1$, the inflection efficiency tends toward zero

[4] Real B and b are again one-to-one. Note in particular: $\lim_{b\to\infty} B \to \infty$ provided $-2\pi/\mathrm{L} < \delta < 2\pi/\mathrm{L}$, $\lim_{b\to 0} B$ = $i\delta(1-e^{i\delta\mathrm{L}/2})/(1+e^{i\delta\mathrm{L}/2}) = \delta Tan[\delta\mathrm{L}/4]$, and $\lim_{b\to-\infty} B = i\delta(1+e^{i\delta\mathrm{L}})/(1-e^{i\delta\mathrm{L}}) = -\delta Cot[\delta\mathrm{L}/2]$. Insisting on *real B* prevents $\lim_{b\to\pm i\delta\mathrm{L}/2} B = \pm i\delta$, for which there is no real trigonometric replacement.

as $b \to -\infty$, but the corresponding inflection age does not increase all the way to L (barely at all as $|z| \to \infty$). Positive b shifts the inflection point higher and (depending on z) earlier: for $-1 \le z \le 1$, increasing b from zero to $ln\left[\frac{1+z^2}{1-z^2}\right]$ implies $\left(s^*, \Phi(s^*/L)\right) = (0, 1)$; for $|z| > 1$, the inflection efficiency tends toward one as $b \to \infty$, but the corresponding inflection age does not decrease all the way to zero (barely at all as $|z| \to \infty$). As in the regular case, curves with the same b but different z always intersect in the interior of the rectangle at $(s, \Phi) = \left(½L, (1+e^{-b})^{-1}\right)$. Further, larger absolute values of z make inflected schedules curve more sharply; but now as z grows very large, b defines a "shelf" form with efficiency $\Phi \approx (1+e^{-b})^{-1}$ for the bulk of the asset's life. Two more graphs will help. First consider:



which collects a family of reclining-chair curves with a common z-value. Unlike the Regular Trapped Logistic, the Irregular form cannot avoid inflecting somewhere within the empirical lifespan; here no Trapped Geometric "lens" separates curves within, which have real inflection points, from curves without, which do not. The lighter schedule traces out the inflection points for all the Irregular forms when $|z|$=1.5. Next consider:



which shows two clusters of curves, now organized about b rather than a. Note the two curves with a=20 are sharply inflected, essentially parallel, and persistently about one-quarter and three-quarters as efficient as a new asset: this is the “shelf” effect for large a. Note also that the two curves with $a \to 0$ are actually disguised Trapped Harmonic / Hyperbolic functions, with $h \approx -1.72$ for $b = -1$ and $h \approx .63$ for b=1.

Like the other trapped forms, the Irregular Trapped Logistic function is extremely flexible, tracking an arbitrary decay process parsimoniously *to a quadratic approximation*, subject to the (ir)regularity condition $B^2-4AC<0$, which allows both one-hump curves and “two-hump” reclining-chair curves.

Age-Price Functions

Unfortunately, most researchers do not observe the marginal products of a single asset as it ages, but rather a cross-section of second-hand prices of the relatively few assets that happen to be resold. Still, relax the fiction only slightly: suppose the researcher observes the fair resale values of one aging asset. This is just the present discounted value of the future service flows of the asset, from its current age until it expires:

$$\mathrm{q(s)} = \int_{\mathrm{s}}^{\mathrm{L}} e^{-\mathrm{r(u-s)}}\,\Phi(\mathrm{u})\,d\mathrm{u},$$

where r is the positive discount rate. Use Leibniz’ Rule to differentiate through the limits of the integral to return to the efficiency form:

$$\Phi(\mathrm{s}) = r\,\mathrm{q(s)} - \partial\mathrm{q(s)}/\partial\mathrm{s},$$

which is an incomplete user-cost formula ($-\partial \mathrm{q}/\partial t$=0 by assumption). The depreciation rate, $\partial ln\mathrm{q(s)}/\partial\mathrm{s}$, is negative, since q(s) slopes down. In fact, if Φ(s) slopes down, so must q(s). But the converse is not true: there are downward-sloping functions $\tilde{\mathrm{q}}\mathrm{(s)}$ for which $r\tilde{\mathrm{q}}\mathrm{(s)}-\tilde{\mathrm{q}}'\mathrm{(s)}$ may slope up for some s. Differentiating both sides of the (complete) user-cost formula a second time and rearranging shows the compound age-price slope condition:

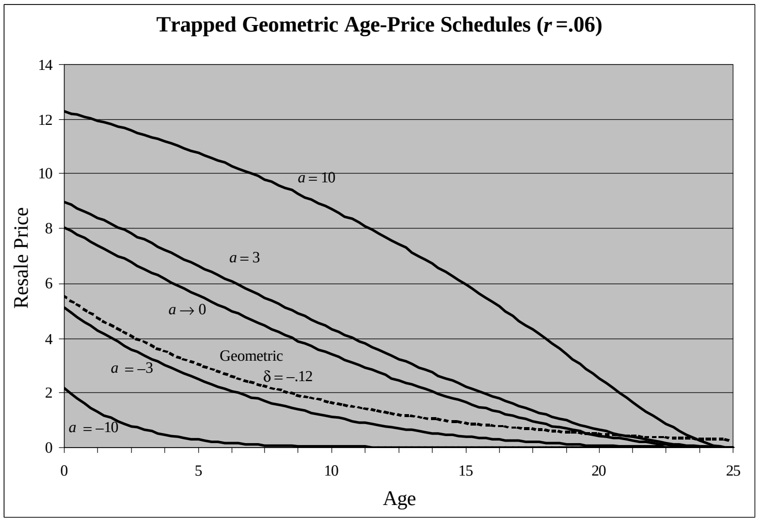
$$\frac{\partial \mathrm{q}}{\partial \mathrm{s}} \le \min\left(0, \frac{\frac{\partial^2 \mathrm{q}}{\partial \mathrm{s}^2} + \frac{\partial^2 \mathrm{q}}{\partial \mathrm{s}\,\partial \mathrm{t}}}{r}\right).$$

Globally convex price functions, such as the Geometric, meet the compound slope condition for all ages (apart from $\partial^2\mathrm{q}/\partial\mathrm{s}\partial t$). But certain $\tilde{\mathrm{q}}\mathrm{(s)}$ that are concave for some ages might, without restrictions, fail the slope condition—and those same restrictions might unduly impinge on the implied age-efficiency shapes. A similar caveat holds at the terminus: if Φ(L)=0 then q(L)=0, but there are $\tilde{\mathrm{q}}\mathrm{(s)}$ such that $r\tilde{\mathrm{q}}\mathrm{(L)}-\tilde{\mathrm{q}}'\mathrm{(L)} \neq 0$ even though $\tilde{\mathrm{q}}\mathrm{(L)}=0$. To avoid these issues, this section starts from the efficiency functions and works up.

The fair resale price integral of a Trapped Geometric asset is straightforward:

$$\mathrm{q(s)} = \int_{\mathrm{s}}^{\mathrm{L}} e^{-r(\mathrm{u-s})}\,\frac{e^{a\mathrm{u/L}} - e^{a}}{1-e^{a}}\,d\mathrm{u} \;=\; \left(\frac{1-e^{r(\mathrm{s-L})}}{r} + \frac{e^{\frac{a}{\mathrm{L}}(\mathrm{s-L})} - e^{r(\mathrm{s-L})}}{\frac{a}{\mathrm{L}} - r}\right)\frac{e^{a}}{e^{a}-1}\,. \qquad (8)$$

Note that q(s), unlike the Trapped Geometric Φ, does not treat *s* and L as a ratio. Age-price functions for the Geometric, straight-line, and one-hoss shay limiting cases all obtain, as well as the limit as $a \to r\mathrm{L}$. Graphs of age-price profiles corresponding to the Trapped Geometric age-efficiency curves drawn above (p. 2) follow. For each schedule, the instantaneous rent on a new asset is normalized to \$1, with r=.06:



The effects of discounting are strong: all but the most strongly concave age-efficiency profiles give way to convex age-price forms, while the age-price profile of a mildly concave asset (e.g., a=3) is roughly linear for the greater part of its working life.

The fair resale price of a Trapped Logistic asset is more complicated. To solve it, first rewrite the integral:

$$\mathrm{q(s)} = \int_{\mathrm{s}}^{\mathrm{L}} e^{-r(\mathrm{u-s})} \frac{e^{a\mathrm{u/L}} - e^{a}}{e^{a\mathrm{u/L}} - e^{a} + e^{\frac{a}{2}-b}(1-e^{a\mathrm{u/L}})} d\mathrm{u} = \frac{e^{r\mathrm{s}}}{e^{\frac{a}{2}-b} - e^{a}} \int_{\mathrm{s}}^{\mathrm{L}} \frac{e^{(\frac{a}{\mathrm{L}}-r)\mathrm{u}} - e^{a-r\mathrm{u}}}{1-\left(\frac{e^{\frac{a}{2}-b}-1}{e^{\frac{a}{2}-b}-e^{a}}\right) e^{a\mathrm{u/L}}} d\mathrm{u}\,.$$

Provided $\left|\frac{e^{\frac{a}{2}-b}-1}{e^{\frac{a}{2}-b}-e^{a}} e^{a\mathrm{u/L}}\right|$ <1 for all u from *s* through L, it is safe to move the denominator of the integral to the numerator as a geometric series. Rearrange as:

$$\mathrm{q(s)} = \left(\frac{e^{r\mathrm{s}}}{e^{\frac{a}{2}-b}-1} - \frac{e^{a+r\mathrm{s}}}{e^{\frac{a}{2}-b}-e^{a}}\right) \int_{\mathrm{s}}^{\mathrm{L}} \sum_{k=1}^{\infty} \left(\frac{e^{\frac{a}{2}-b}-1}{e^{\frac{a}{2}-b}-e^{a}}\right)^{k} e^{(k\frac{a}{\mathrm{L}}-r)\mathrm{u}} d\mathrm{u} - \frac{e^{a+r\mathrm{s}}}{e^{\frac{a}{2}-b}-e^{a}} \int_{\mathrm{s}}^{\mathrm{L}} e^{-r\mathrm{u}} d\mathrm{u}\,,$$

and then work out the definite integral:

$$\mathrm{q(s)} = \left\{ \frac{1-e^{r(\mathrm{s-L})}}{r} + \left(\frac{e^{\frac{a}{2}}-e^{-\frac{a}{2}}}{e^{b}-e^{\frac{a}{2}}}\right) \sum_{k=1}^{\infty} \left[\left(\frac{e^{b}-e^{\frac{a}{2}}}{e^{b}-e^{-\frac{a}{2}}}\right)^{k} \frac{e^{k\frac{a}{\mathrm{L}}(\mathrm{s-L})} - e^{r(\mathrm{s-L})}}{k\frac{a}{\mathrm{L}}-r} \right] \right\} \frac{e^{a}}{e^{a}-e^{\frac{a}{2}-b}}\,, \qquad (9.1)$$

which resembles the Trapped Geometric price integral but features two alternating series.[5] To find out

[5] Canceling powers of $e^{b}-e^{a/2}$ and taking the limit as $b \to a/2$ returns form (8), as $\lim\,(e^{b}-e^{a/2})^{0}=1$.

whether the expression converges, rewrite it using Gauss' hypergeometric function[6]:

(9.2)

$$q(s) = \left\{\frac{1-e^{r(s-L)}}{r} - \left(\frac{e^{\frac{a}{2}}-e^{-\frac{a}{2}}}{e^{b}-e^{-\frac{a}{2}}}\right)\left[{}_2F_1\!\left(1,-\frac{rL}{a},1-\frac{rL}{a},\frac{e^{b}-e^{\frac{a}{2}}}{e^{b}-e^{-\frac{a}{2}}}e^{\frac{a}{L}(s-L)}\right) - e^{r(s-L)}\,{}_2F_1\!\left(1,-\frac{rL}{a},1-\frac{rL}{a},\frac{e^{b}-e^{\frac{a}{2}}}{e^{b}-e^{-\frac{a}{2}}}\right)\right]\Big/ r\right\}\frac{1}{1-e^{\frac{a}{2}-b}}\,.$$

A hypergeometric function converges if the absolute value of its fourth argument—here, $\left|\frac{e^{b}-e^{\frac{a}{2}}}{e^{b}-e^{-\frac{a}{2}}}e^{\frac{a}{L}(s-L)}\right|$ for any non-negative *s* up through L—is less than one. But this merely restates the condition already assumed, by which the denominator of the integrand was transferred to the numerator as a geometric series.

On the other hand, if $\left|\frac{e^{\frac{a}{2}-b}-1}{e^{\frac{a}{2}-b}-e^{a}}e^{au/L}\right|$ >1 for all u from *s* through L, then multiply the numerator and denominator of the integrand by $\frac{e^{\frac{a}{2}-b}-e^{a}}{e^{\frac{a}{2}-b}-1}e^{-au/L}$ in order to apply the geometric series. Rearranging gives:

$$q(s) = \frac{e^{rs}}{e^{-\frac{a}{2}-b}-e^{-a}}\int_s^L \frac{e^{(-\frac{a}{L}-r)u}-e^{-a-ru}}{1-\left(\frac{e^{-\frac{a}{2}-b}-1}{e^{-\frac{a}{2}-b}-e^{-a}}e^{-au/L}\right)}du\,,$$

which is algebraically the same as before, only with –*a* instead of *a*. The solution is therefore expression (9.1) or (9.2) again, but with the sign of *a* switched.

Finally, suppose: $\left|\frac{e^{\frac{a}{2}-b}-1}{e^{\frac{a}{2}-b}-e^{a}}e^{au/L}\right|$ <1 for all non-negative u from *s* until *s**, $\left|\frac{e^{\frac{a}{2}-b}-1}{e^{\frac{a}{2}-b}-e^{a}}e^{au/L}\right|$ >1 for u from *s** until L, and $\frac{e^{\frac{a}{2}-b}-1}{e^{\frac{a}{2}-b}-e^{a}}e^{as^*/L}=-1$.[7] Then rearrange the integrand in two pieces:

$$\begin{aligned} q(s) = &\left(\frac{e^{rs}}{e^{\frac{a}{2}-b}-1}-\frac{e^{a+rs}}{e^{\frac{a}{2}-b}-e^{a}}\right)\int_s^{s^*}\sum_{k=1}^{\infty}\left(\frac{e^{\frac{a}{2}-b}-1}{e^{\frac{a}{2}-b}-e^{a}}\right)^k e^{(k\frac{a}{L}-r)u}du-\frac{e^{a+rs}}{e^{\frac{a}{2}-b}-e^{a}}\int_s^{s^*}e^{-ru}du \\ &+\left(\frac{e^{rs}}{e^{\frac{-a}{2}-b}-1}-\frac{e^{-a+rs}}{e^{\frac{-a}{2}-b}-e^{-a}}\right)\int_{s^*}^{L}\sum_{k=1}^{\infty}\left(\frac{e^{\frac{-a}{2}-b}-1}{e^{\frac{-a}{2}-b}-e^{-a}}\right)^k e^{(k\frac{-a}{L}-r)u}du-\frac{e^{-a+rs}}{e^{\frac{-a}{2}-b}-e^{-a}}\int_{s^*}^{L}e^{-ru}du\,,\end{aligned}$$

and solve:

(9.3)

$$q(s) = \left\{\frac{e^{r(s-s^*)}-e^{r(s-L)}}{r}+\left(\frac{e^{-\frac{a}{2}}-e^{\frac{a}{2}}}{e^{b}-e^{-\frac{a}{2}}}\right)\sum_{k=1}^{\infty}\left[\left(\frac{e^{b}-e^{-\frac{a}{2}}}{e^{b}-e^{\frac{a}{2}}}\right)^k\frac{e^{-k\frac{a}{L}(s^*-L)+r(s-s^*)}-e^{r(s-L)}}{-k\frac{a}{L}-r}\right]\right\}\frac{e^{-a}}{e^{-a}-e^{-\frac{a}{2}-b}}\ \ldots$$

---

[6] ${}_2F_1(n_1,n_2,d,x)=1+\frac{n_1n_2}{d}x+\frac{n_1(n_1+1)n_2(n_2+1)}{2!d(d+1)}x^2+\frac{n_1(n_1+1)(n_1+2)n_2(n_2+1)(n_2+2)}{3!d(d+1)(d+2)}x^3+\ldots$ See Abramowitz and Stegun [1964, reprinted with corrections] Chapter 15.

[7] Calling the boundary age *s** is no accident, for the same condition rearranged identifies the inflection age, which is real if $-|a|/2 \le b \le |a|/2$ (see p. 4). For ages from 0 through finite L, $e^{as^*/L}(e^{\frac{a}{2}-b}-1)/(e^{\frac{a}{2}-b}-e^{a}) \ne 1$.

$$\ldots + \left\{ \frac{1 - e^{r(s-s^*)}}{r} + \left(\frac{e^{\frac{a}{2}} - e^{-\frac{a}{2}}}{e^{b} - e^{\frac{a}{2}}} \right) \sum_{k=1}^{\infty} \left[\left(\frac{e^{b} - e^{\frac{a}{2}}}{e^{b} - e^{-\frac{a}{2}}} \right)^{k} \frac{e^{k\frac{a}{L}(s-L)} - e^{k\frac{a}{L}(s^*-L) + r(s-s^*)}}{k\frac{a}{L} - r} \right] \right\} \frac{e^{a}}{e^{a} - e^{\frac{a}{2} - b}} \,.$$

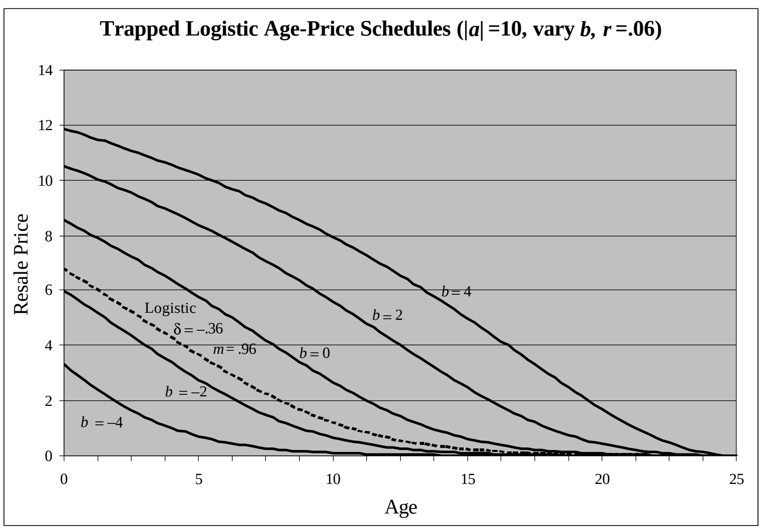
This double form reduces to the "*a*" parametrization of expression (9.1) when s*=L, but the "–*a*" form when s*=0. The hypergeometric double form is:

(9.4)

$$q(s) = \left\{ \frac{e^{r(s-s^*)} - e^{r(s-L)}}{r} - \left(\frac{e^{-\frac{a}{2}} - e^{\frac{a}{2}}}{e^{b} - e^{\frac{a}{2}}} \right) \left[e^{r(s-s^*)} {}_2F_1\left(1, \frac{rL}{a}, 1 + \frac{rL}{a}, \frac{e^{b} - e^{-\frac{a}{2}}}{e^{b} - e^{\frac{a}{2}}} e^{-\frac{a}{L}(s^*-L)}\right) - e^{r(s-L)} {}_2F_1\left(1, \frac{rL}{a}, 1 + \frac{rL}{a}, \frac{e^{b} - e^{-\frac{a}{2}}}{e^{b} - e^{\frac{a}{2}}}\right) \right] \Big/ r \right\} \frac{1}{1 - e^{-\frac{a}{2} - b}}$$

$$+ \left\{ \frac{1 - e^{r(s-s^*)}}{r} - \left(\frac{e^{\frac{a}{2}} - e^{-\frac{a}{2}}}{e^{b} - e^{-\frac{a}{2}}} \right) \left[{}_2F_1\left(1, -\frac{rL}{a}, 1 - \frac{rL}{a}, \frac{e^{b} - e^{\frac{a}{2}}}{e^{b} - e^{-\frac{a}{2}}} e^{\frac{a}{L}(s-L)}\right) - e^{r(s-s^*)} {}_2F_1\left(1, -\frac{rL}{a}, 1 - \frac{rL}{a}, \frac{e^{b} - e^{\frac{a}{2}}}{e^{b} - e^{-\frac{a}{2}}} e^{\frac{a}{L}(s^*-L)}\right) \right] \Big/ r \right\} \frac{1}{1 - e^{\frac{a}{2} - b}} \,.$$

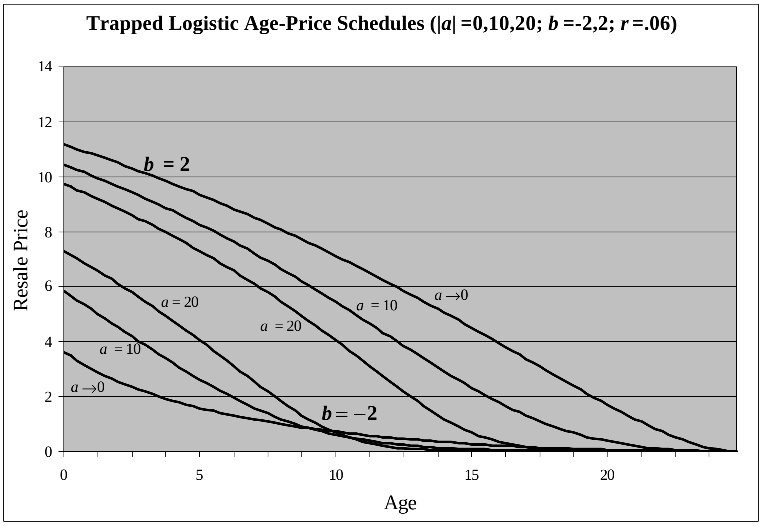
The one-sided cases already establish the convergence of each hypergeometric series for u≠s*. At the inflection age—i.e., when $\frac{e^{b} - e^{-\frac{a}{2}}}{e^{b} - e^{\frac{a}{2}}} e^{-\frac{a}{L}(s^*-L)} = \frac{e^{b} - e^{\frac{a}{2}}}{e^{b} - e^{-\frac{a}{2}}} e^{\frac{a}{L}(s^*-L)} = -1$—the first and fourth hypergeometric series in (9.4) reduce to sums of digamma functions: ${}_2F_1\left(1, \frac{rL}{a}, 1 + \frac{rL}{a}, -1\right) = \frac{rL}{2a}\left[\psi\left(\frac{1}{2} + \frac{rL}{2a}\right) - \psi\left(\frac{rL}{2a}\right)\right]$ and ${}_2F_1\left(1, -\frac{rL}{a}, 1 - \frac{rL}{a}, -1\right) = \frac{rL}{2a}\left[\psi\left(-\frac{rL}{2a}\right) - \psi\left(\frac{1}{2} - \frac{rL}{2a}\right)\right]$, which are well defined unless the argument of either ψ function is zero or a negative integer.[8] Two graphs of age-price profiles corresponding to the Trapped Logistic age-efficiency curves drawn above (p. 5) follow. Again the instantaneous rent of a new asset is normalized to $1, while r=.06 and L=25. Curves in the first set:



resemble the Trapped Geometric age-price profiles. This is not surprising for the "outside" curves (e.g., b=4 and –4), whose underlying age-efficiency schedules are nearly Trapped Geometric ($b \to \pm a/2$). But it is noteworthy for the "inside" curves, whose underlying age-efficiency schedules are sharply inflected.

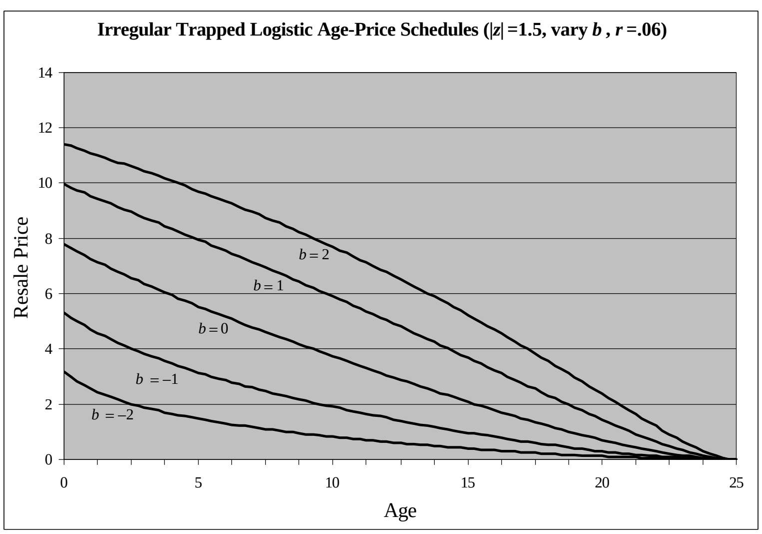
[8] But if either $\pm\frac{rL}{2a}$ or $\frac{1}{2} \pm \frac{rL}{2a}$ is zero or a negative integer, the original hypergeometric function reduces to a polynomial. The bottom line: the price integral is convergent at the interior inflection age s*.

The second set:

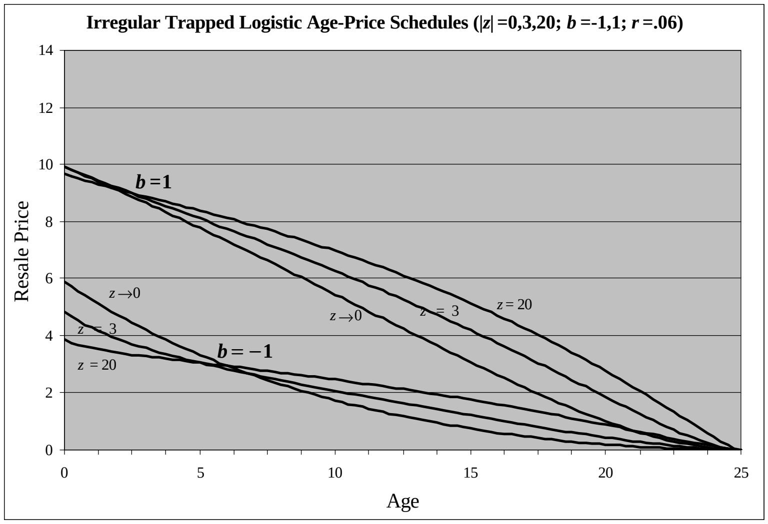


tells a similar story. Among "two-hump" efficiency forms, only the "waterfall" cases (i.e., *a*=20) retain much price-side concavity. The top curve (i.e., $a \rightarrow 0$ and *b*=2) is concave for most of its career because it corresponds to a Hyperbolic / Trapped Harmonic "one-hump" efficiency profile not far from the one-hoss shay ($h \approx .86$).

The fair resale price of an Irregular Trapped Logistic asset is more complicated still, and as I have abused the conference deadline enough already, I do not present any analytical results. Nonetheless two graphs of age-price profiles corresponding to the Irregular Trapped Logistic age-efficiency curves drawn above (p. 9) follow. The bookkeeper faced with the first set:



could be forgiven for applying the straightline formula as a rule of thumb to the asset (*z*=1.5, *b*=1). Assets in the second graph (below) keep their midlife value a bit better than Trapped Geometric assets, but the effect is noticeable only when the underlying efficiency curves are sharply inflected "shelves" (i.e. *z*=20):



The similarity of most individual age-price profiles to schedules derived from Trapped Geometric efficiency forms should give productivity economists pause. Unless price data for a single asset are available in a nearly continuous stream, the compacted nature of the information they contain on current and future marginal products pretty well prevents regressions on used prices from uncovering the fine details of the asset's decay: *average* curvature and the lifespan, little more.[9] But the empirical researcher who has made peace with these limitations can always used the Trapped Geometric price function.

<u>Shortcuts</u>

The difficulty of the previous section might tempt the researcher to find a generically flexible downward-sloping schedule that finishes at zero, declare it a price function, and hope for the best. The algebraic forms of the Trapped efficiency functions slope downward, assume a variety of shapes, and hit zero: maybe, suitably restricted, they could be pressed into service as age-price functions. Consider for example the "Putative Trapped Geometric" age-price function:

$$\tilde{q}(s) = b\,\frac{e^{as/L} - e^{a}}{1 - e^{a}},$$

which has implied efficiency:

$$\tilde{q}(s)\left(r - \frac{a}{L}\right) - b\,\frac{a}{L}\,\frac{e^{a}}{1 - e^{a}},$$

where $b = \dfrac{L(1-e^{a})}{rL(1-e^{a}) - a}$ normalizes initial efficiency to 1. The price form $\tilde{q}(s)$ always slopes down, so restricting $a<rL$ guarantees the same for efficiency. But while $\tilde{q}(L)=0$, the efficiency side is always positive except in the Geometric limit. Perhaps applying "the trap" to the implied efficiency function, and then integrating back up for a consistent price form, would fix things. But observe:

[9] The problem is worse in actual cross sections, where "lifespan" is not a single parameter but a random variable with different realizations (not observed by the economist) across assets.

$$\frac{\left[\tilde{\mathrm{q}}(\mathrm{s})\left(r-\frac{a}{\mathrm{L}}\right)-b\,\frac{a}{\mathrm{L}}\,\frac{e^{a}}{1-e^{a}}\right]-\left[\tilde{\mathrm{q}}(\mathrm{L})\left(r-\frac{a}{\mathrm{L}}\right)-b\,\frac{a}{\mathrm{L}}\,\frac{e^{a}}{1-e^{a}}\right]}{\left[\tilde{\mathrm{q}}(0)\left(r-\frac{a}{\mathrm{L}}\right)-b\,\frac{a}{\mathrm{L}}\,\frac{e^{a}}{1-e^{a}}\right]-\left[\tilde{\mathrm{q}}(\mathrm{L})\left(r-\frac{a}{\mathrm{L}}\right)-b\,\frac{a}{\mathrm{L}}\,\frac{e^{a}}{1-e^{a}}\right]}=\frac{e^{as/\mathrm{L}}-e^{a}}{1-e^{a}} \quad \ldots!$$

—which is just the original Trapped Geometric efficiency function. The implied efficiency of a "Putative Trapped Logistic" age-price function—i.e., first set $\tilde{\mathrm{q}}(\mathrm{s})$ equal to expression (6) times β, then multiply by *r* and subtract the derivative—also would remain positive at *s*=L, since $\tilde{\mathrm{q}}'(\mathrm{L})\neq 0$. Applying the trap in that case would not reduce implied efficiency to the original Trapped Logistic profile, so there is hope for integrating back up to a consistent price form, but I have not examined the restrictions necessary to make both price and efficiency decline with age to zero.

The well known Box-Cox age-(and time)-price function:

$$\frac{\mathrm{q}^{q_1}-1}{q_1}=a+b\,\frac{\mathrm{s}^{q_2}-1}{q_2}+g\,\frac{\mathrm{t}^{q_3}-1}{q_3} \tag{10}$$

has been employed in a number of depreciation studies owing to its great flexibility. The form can fit straightline ($\theta_1=\theta_2=\theta_3=1$), Geometric ($\theta_1=0$, $\theta_2=\theta_3=1$), log-log ($\theta_1=\theta_2=\theta_3=0$), and backward-S ($\theta_1<1$, $\theta_2>1$) price curves, as well as many others. Implied efficiency is:

$$\Phi(\mathrm{s,t}) = \mathrm{q}\,[r-(\beta\,\mathrm{s}^{\theta_2-1}+\gamma\,\mathrm{t}^{\theta_3-1})\mathrm{q}^{-\theta_1}]. \tag{11}$$

Apart from *t* (i.e., for $\gamma=0$) and for θ_1 strictly positive, one may constrain α or β to enforce q(L)=0; the same constraint forces implied efficiency to zero when *s*=L, so the complications of trapping are not needed. The price function slopes down with age provided $\beta<0$. The slope of Φ(s,t) with age is:

$$\partial\Phi/\partial\mathrm{s} = r\,\partial\mathrm{q}/\partial\mathrm{s} - \beta(\theta_2-1)\,\mathrm{s}^{\theta_2-2}\mathrm{q}^{1-\theta_1} - (1-\theta_1)(\beta\,\mathrm{s}^{\theta_2-1}+\gamma\,\mathrm{t}^{\theta_3-1})\mathrm{q}^{-\theta_1}\,\partial\mathrm{q}/\partial\mathrm{s}$$

$$= \frac{\partial\mathrm{q}}{\partial\mathrm{s}}\,[r+(\theta_1-1)(\frac{\partial ln\,\mathrm{q}}{\partial\mathrm{s}}+\frac{\partial ln\,\mathrm{q}}{\partial\mathrm{t}})+\frac{1-q_2}{\mathrm{s}}].$$

Given $\partial\mathrm{P}/\partial\mathrm{s} < 0$, the slope of Φ is negative if the bracketed terms are positive. So long as $r>0$ and $\partial ln\mathrm{q}/\partial\mathrm{s} + \partial ln\mathrm{q}/\partial\mathrm{t} < 0$—i.e., the prices of used assets fall over time, from whatever combination of age and time effects—then $\theta_1\leq 1$ and $\theta_2\leq 1$ are together sufficient for the correct sign. Together they lock in an accelerated depreciation pattern, although this is not strictly necessary: with a large enough *r* or a positive enough $\partial ln\mathrm{q}/\partial\mathrm{t}$ (i.e., $\gamma>0$), either θ_1 or θ_2 (or both) may exceed unity without harm, at least when *s* is large.

But when *s* is small, things are different, because it is the *reciprocal of s* that appears in ∂Φ/∂s. Now if $\theta_2>1$ there will be some positive *s* below which the bracketed terms sum to a negative number: implied efficiency will pop up when *s* is near zero. On the other hand, $\theta_2<1$ makes the bracketed terms *very* positive for small *s*: implied efficiency will fall faster than geometric when *s* is near zero. A global remedy—forcing θ_2=1 and relying on θ_1 for curvature —removes some of the Box-Cox form's flexibility. A local remedy—renumbering age, so that *s* never gets too small[10], and then perhaps bumping up *r* after the fact to lock in the proper slope—would attenuate the problem, but it would be advantageous to estimate *r* directly in the used-price regression. One would also want to test the sensitivity of implied age-efficiency patterns to various renumberings.

Extensions

So far this paper has found one functional form, the Trapped Geometric, that accommodates a variety of one-hump age-efficiency profiles, tracks any well-behaved efficiency schedule to a linear approximation, and has an econometrically amenable age-price form. The paper also identified the Trapped Logistic forms (regular and irregular) as quadratic approximations to well-behaved age-efficiency functions, but the implied age-price profiles are difficult to implement (numerical integration, such as Simpson's method, remains a possibility). Fortunately(?), age-price curves for most two-hump efficiency schedules are not very different from those for one-hump schedules: the accumulation of discounted future efficiencies into the resale price allows the researcher a measure of sloppiness. Nonetheless some two-hump efficiency profiles are curved so sharply that their implied price schedules will differ noticeably from the prices of one-hump or mildly two-humped efficiency profiles. Other two-hump efficiency patterns may inflect outside the upper-left or lower-right quarters of the age-efficiency rectangle, so the approximating powers of Trapped Logistic efficiency forms will falter.

A practical answer to these problems is to apply two different Trapped Geometric functions to different sections of the same curve, taking care to connect the two functions smoothly. Suppose the link occurs at (s, Φ) = (H, V), so H represents the "horizontal" coordinate of the connection ($0 \le H \le L$), while V is the "vertical" coordinate ($1 \ge V \ge 0$). Next let $D(s \le H)$ be a dummy variable set to 1 for all ages at or below H but 0 for all ages above. Then a two-part Trapped Geometric approximation to a two-hump efficiency curve is:

$$\Phi(s, H, L) = D(s\le H)\left(\frac{e^{as/H}-e^{a}}{1-e^{a}}(1-V)+V\right) + [1-D(s\le H)]\left(\frac{e^{b\frac{s-H}{L-H}}-e^{b}}{1-e^{b}}V\right). \qquad (12)$$

Consider the two pieces separately. The first part, $\frac{e^{as/H}-e^{a}}{1-e^{a}}(1-V)+V$, equals 1 when s=0 and falls to V as s=H; any further plunge is arrested by the dummy. The second part, $\frac{e^{b\frac{s-H}{L-H}}-e^{b}}{1-e^{b}}V$, starts at V when s=H and declines to 0 as s=L; the dummy prevents efficiencies for $s \le$H. The slope of the first part is $\frac{a}{H}\frac{e^{a}}{1-e^{a}}(1-V)$ when s=H; the slope of the second part at the same age is: $\frac{b}{L-H}\frac{V}{1-e^{b}}$. To smooth the connection, equate slopes and solve for V:

[10] See Table 5 in Hulten and Wykoff [1981b, p. 387], where 1-year-old structures are termed "new."

$$V = \frac{\frac{a}{H}\frac{e^a}{1-e^a}}{\frac{a}{H}\frac{e^a}{1-e^a} + \frac{b}{(L-H)(1-e^b)}}. \tag{13}$$

Note that V always lies between 0 and 1, and that $\lim_{a \text{ or } b\to\infty} V = \lim_{H\to 0} V = 1$ and $\lim_{a \text{ or } b\to-\infty} V = \lim_{H\to L} V = 0$. But the smoothness restriction does not prevent the (H, V) joint from occurring anywhere within the age-efficiency rectangle.[11] If $b = -a$, then V simplifies to $\frac{L-H}{L}$, restricting connection points to the straightline diagonal.

The age-price form that corresponds to (12) is:

(14)

$$q(s) = D(s \le H)\left[\left(\frac{1-e^{r(s-H)}}{r} + \frac{e^{\frac{a}{H}(s-H)} - e^{r(s-H)}}{\frac{a}{H}-r}\right)\frac{e^a(1-V)}{e^a-1} + \frac{1-e^{r(s-H)}}{r}V + \left(\frac{e^{r(s-H)}-e^{r(s-L)}}{r} + \frac{e^{r(s-H)-b}-e^{r(s-L)}}{\frac{b}{L-H}-r}\right)\frac{e^bV}{e^b-1}\right]$$

$$+[1-D(s\le H)]\left(\frac{1-e^{r(s-L)}}{r} + \frac{e^{b\frac{s-L}{L-H}} - e^{r(s-L)}}{\frac{b}{L-H}-r}\right)\frac{e^bV}{e^b-1}.$$

This is substantially nonlinear, particularly given V, but not beyond the reach of careful estimation. A grid search is needed in any event since H is constrained by the top age of the domain of the dummy variable. An actual regression of used prices would need to multiply (14) by an arbitrary constant, since the user cost of a new asset may differ from 1. Finally, the parameters may be enhanced by vintage effects—e.g., replace L by $L_o+L_v(v-v_o)$—which is consistent with treating the lifespan and curvature as technical rather than behavioral features. Although one may then substitute for time effects by the familiar identity $t=s+v$, the resulting form and its derived efficiency might not be well behaved.

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U.S. Bureau of Labor Statistics
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[11] Even with the slope restriction, the two-part Trapped Geometric is not as smooth as the Trapped Logistic because the *second* derivatives are not continuous at the meeting point. A Trapped Geometric form never has a second derivative of zero, so the discussion here is of connection, not inflection.

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