Preliminary and Incomplete

Matching with Heterogeneity

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June 2000

Prepared for presentation at the NBER Summer Institute 2000

1. INTRODUCTION

Individuals are constrained by the number of hours in a day that they have to work with. Also, employers are often constrained by the number of jobs that they have to fill. These facts make the analysis of equilibria with capacity constraints, developed by Peters (1984), particularly relevant to the labor market. Recent work by Montgomery (1991), Burdett, Shi, and Wright (2000), Julien, Kennes, and King (2000a,b), Shi (1999), and Shimer (2000) has explored the implications of this insight. One major implication is that, when buyers randomize in their decision about which seller to approach, the equilibrium matching function generated by this process in large economies shares many properties with the matching function commonly used in macroeconomics (for example, Pissarides (1990)). Thus, these "directed search" models can be used to analyse factors that influence this matching process.

The existence of heterogeneity in a matching problem can either worsen or improve the problem of coordination, depending on the nature of the heterogeneity. In this paper, we explore this relationship in a simple competing-auction model with two-sided heterogeneity, two buyers, and two sellers. We demonstrate that, in the presence of either strict supermodularity or strict submodularity, the unique equilibrium has buyers playing pure strategies and the equilibrium allocation is efficient. With strict supermodularity, the equilibrium exhibits positive assortative matching, and with strict submodularity it exhibits negative assortative matching. However, when matches are both supermodular and submodular then multiple equilibria exist. In particular, in this case, a coordination problem and equilibria with mixing exist. Several interesting special cases have this property: homogeneous agents, one-sided heterogeneity, and additive two-sided heterogeneity. In this setting, the mixing equilibrium is inefficient but, given the coordination friction, is constrained-efficient in the sense that a constrained planner would assign the same probabilities for buyer locations if the planner were unable to coordinate. Thus, if heterogeneity is either strictly supermodular or submodular, then it can act as a "coordination device" (as argued by Coles and Eeckhout (1999), assuming supermodularity) in the sense that it rules out the mixing equilibrium present in the homogenous agent case. However, if matching is both submodular and supermodular, then the coordination problem is strictly increasing in the magnitude of heterogeneity.

2. THE MODEL

The market consists of 2 buyers and 2 sellers, all of whom are risk neutral, and seek to maximize their expected payoffs. Each seller has one unit of a good to sell. If she sells at price p then her payoff is p. If she is unable to sell, then her payoff is zero. Each buyer wants to purchase one unit of the good, and is willing to pay up to his valuation of the good. The valuation that buyer j places on the good offered by seller i is denoted S_{ij} for $i, j \in \{1,2\}$. We place the following restrictions on S:

$$S_{11} \geq S_{12}, S_{21} \geq S_{22} \tag{2.1}$$

If buyer j purchases from seller i at price p_{ij}, his payoff is $S_{ij} - p_{ij}$. If he is unable to buy then his payoff is zero. Each buyer has only one opportunity to buy, and must choose only one seller to visit.

The order of play is as follows. First, sellers announce reserve prices s_{ij} (in general, these are different for the different buyers -- we allow for price discrimination). Second, after observing s, buyers choose which seller to approach. Finally, once buyers have been allocated to sellers, goods are sold according to a bidding game. Equilibria are solved through backwards induction.

2.1 The Bidding Game

Let w_i denote the payoff for seller i. In the bidding game, this is a function of the allocation of buyers:

$$w_i = \begin{cases} 0 & if \quad no \quad buyers\ visit \\ s_{i1} & if \quad only \quad buyer \quad 1 \quad visits \\ s_{i2} & if \quad only \quad buyer \quad 2 \quad visits \\ S_{i2} & if \quad both \quad buyers\ visit \end{cases} \tag{2.2}$$

Clearly, if no buyers visit seller *i*, then her payoff will be zero. If only one approaches, then the seller sells at the reserve price s_{ij}. If both buyers visit, then the bidding game determines that the good is sold to the buyer with the highest valuation (buyer 1) at the price of the second-highest valuation S_{i2}.

2.2 Buyers' Location Choices

Given the reserve price matrix s, buyer *j*'s payoff from approaching seller *i* is given by:

$$R_{ij} = \begin{cases} S_{ij} - s_{ij} & if \quad alone \\ S_{ij} - S_{i2} & if \quad not \quad alone \end{cases} \tag{2.3}$$

Buyer payoffs in this location subgame can be represented in the normal form:

Buyer 2

Buyer 1		Seller 1	Seller 2
	Seller 1	$S_{11} - S_{12}$, 0	$S_{11} - s_{11}$, $S_{22} - s_{22}$
	Seller 2	$S_{21} - s_{21}$, $S_{12} - s_{12}$	$S_{21} - S_{22}$, 0

Table 1

Let P_{ij} denote the probability that buyer *j* assigns to visiting seller *i*. Buyer *j*'s expected payoff from visiting seller *i* is given by:

$$\Pi_{ij} = (1 - P_{i,-j})(S_{ij} - s_{ij}) + P_{i,-j}(S_{ij} - S_{i2}) \tag{2.4}$$

The following 9 distinct possible cases exist for buyer 1:

1. $S_{11} - S_{12} > S_{21} - s_{21}$ and $S_{11} - s_{11} > S_{21} - S_{22}$
2. $S_{11} - S_{12} > S_{21} - s_{21}$ and $S_{11} - s_{11} = S_{21} - S_{22}$
3. $S_{11} - S_{12} > S_{21} - s_{21}$ and $S_{11} - s_{11} < S_{21} - S_{22}$
4. $S_{11} - S_{12} = S_{21} - s_{21}$ and $S_{11} - s_{11} > S_{21} - S_{22}$
5. $S_{11} - S_{12} = S_{21} - s_{21}$ and $S_{11} - s_{11} = S_{21} - S_{22}$
6. $S_{11} - S_{12} = S_{21} - s_{21}$ and $S_{11} - s_{11} < S_{21} - S_{22}$
7. $S_{11} - S_{12} < S_{21} - s_{21}$ and $S_{11} - s_{11} > S_{21} - S_{22}$
8. $S_{11} - S_{12} < S_{21} - s_{21}$ and $S_{11} - s_{11} = S_{21} - S_{22}$
9. $S_{11} - S_{12} < S_{21} - s_{21}$ and $S_{11} - s_{11} < S_{21} - S_{22}$

Restricting reserve prices to be no greater than valuations $(s_{ij} \le S_{ij})$, there are 4 distinct cases for buyer 2:

A. $S_{22} > s_{22}$ and $S_{12} > s_{12}$

B. $S_{22} > s_{22}$ and $S_{12} = s_{12}$

C. $S_{22} = s_{22}$ and $S_{12} > s_{12}$

D. $S_{22} = s_{22}$ and $S_{12} = s_{12}$

Lemma 1: The buyers' subgame has the following equilibria in pure strategies:

Case	Up, Left	Up, Right	Down, Left	Down, Right
1A		yes		
1B		yes		
1C	yes	yes		
1D	yes	yes		
2A		yes		
2B		yes		yes
2C	yes	yes		
2D	yes	yes		yes
3A				
3B				yes
3C	yes			
3D	yes			yes
4A		yes	yes	
4B		yes	yes	
4C	yes	yes	yes	
4D	yes	yes	yes	
5A		yes	yes	
5B		yes	yes	yes
5C	yes	yes	yes	
5D	yes	yes	yes	yes
6A			yes	
6B			yes	yes
6C	yes		yes	
6D	yes		yes	yes
7A		yes	yes	
7B		yes	yes	
7C		yes	yes	
7D		yes	yes	
8A		yes	yes	
8B		yes	yes	yes
8C		yes	yes	
8D		yes	yes	yes
9A			yes	
9B			yes	yes
9C			yes	
9D			yes	yes

Proof: Follows from exhaustive examination of Table 1. QED

Notice that, in every case except 3A, some equilibrium exists in this subgame.

Lemma 2: Wherever mixing equilibria exist, visit probabilities are given by:

$$P_{11} = \frac{S_{12} - s_{12}}{S_{22} + S_{12} - s_{22} - s_{12}}$$

$$P_{12} = \frac{S_{22} - s_{22}}{S_{22} + S_{12} - s_{22} - s_{12}}$$

$$P_{21} = \frac{S_{11} + S_{22} - S_{21} - s_{11}}{S_{22} + S_{12} - s_{22} - s_{12}}$$

$$P_{22} = \frac{S_{21} + S_{12} - S_{11} - s_{21}}{S_{22} + S_{12} - s_{22} - s_{12}}$$

Proof: Buyers choose their probabilities to be indifferent between which seller is visited. Setting $\Pi_{1j} = \Pi_{2j}$, j = 1, 2, in (2.4), one obtains the above values for P. QED

2.3 Sellers' Reserve Price Decisions

We now turn to consider sellers' optimal choices of s. We start by considering equilibria in pure strategies.

Proposition 1: When considering equilibria in pure strategies:

a) No equilibria exist where both buyers visit the same seller.

b) Equilibria with positive assortative matching exist if and only if match valuations are supermodular:

$$S_{11} + S_{22} \geq S_{12} + S_{21} \tag{2.5}$$

c) Equilibria with negative assortative matching exist if and only if match valuations are submodular:

$$S_{11} + S_{22} \leq S_{12} + S_{21} \tag{2.6}$$

d) Positive assortative and negative assortative matching equilibria co-exist if and only if match valuations are both supermodular and submodular:

$$S_{11} + S_{22} = S_{12} + S_{21} \tag{2.7}$$

Proof: (Sketch)

a) Examination of the table in Lemma 1 reveals that the following restrictions must be imposed for (Up, Left) to be an equilibrium of the buyers' subgame:

$$s_{21} \geq S_{21} - (S_{11} - S_{12}) \quad , \quad s_{22} = S_{22}$$

For all possible ranges of $s_1^* = (s_{11}^*, s_{12}^*)$, one can find a vector $s_2 = (s_{21}, s_{22})$ which violates the above restrictions and which yields a strictly larger payoff for seller 2 in all equilibria of the (s_1^*, s_2) subgame. All possible (Down, Right) equilibria are ruled out similarly.

b) Consider the following candidate equilibrium:

$$P_{11}^* = 1, \quad P_{22}^* = 1$$
$$s_{11}^* = S_{11} - (S_{21} - S_{22}), \quad s_{12}^* = S_{12}, \quad s_{21}^* = S_{21} - (S_{11} - S_{12}), \quad s_{22}^* = S_{22}$$

This is an equilibrium of the buyers' subgame (case 5D). Also, for every s_1, one can demonstrate that there exists some equilibrium of the (s_1, s_2^*) subgame where seller 1 is no better off, iff S is supermodular. Similarly, for every s_2, one can demonstrate that there exists some equilibrium of the

(s_1^*, s_2) subgame where seller 2 is no better off. Thus, (s_1^*, s_2^*) is a subgame perfect equilibrium. Similarly, one can find subgame perfect equilibria for all

$P_{11}^* = 1$, $P_{22}^* = 1$

$s_{11}^* = S_{11} - (S_{21} - S_{22})$, $s_{12}^* \in [0, S_{12}]$, $s_{21}^* \in [0, S_{21}]$, $s_{22}^* = S_{22}$

iff S is supermodular.

c) Consider the following candidate equilibrium:

$P_{11}^* = 0$, $P_{22}^* = 0$

$s_{11}^* = S_{11}$, $s_{12}^* = S_{12}$, $s_{21}^* = S_{21} - (S_{11} - S_{12})$, $s_{22}^* = S_{22}$

This is an equilibrium of the buyers' subgame (case 6D). Also, for every s_1, one can demonstrate that there exists some equilibrium of the (s_1, s_2^*) subgame where seller 1 is no better off. Similarly, for every s_2, one can demonstrate that there exists some equilibrium of the (s_1^*, s_2) subgame where seller 2 is no better off iff S is submodular. Thus, (s_1^*, s_2^*) is a subgame perfect equilibrium. Similarly, one can find subgame perfect equilibria for all

$P_{11}^* = 0$, $P_{22}^* = 0$

$s_{11}^* \in [0, S_{11}]$, $s_{12}^* = S_{12}$, $s_{21}^* = S_{21} - (S_{11} - S_{12})$, $s_{22}^* \in [0, S_{22}]$

iff S is submodular.

d) Follows immediately from (b) and (c).

Discussion

Notice that positive (negative) assortative matching maximizes aggregate surplus when match valuations are supermodular (submodular). Thus, the pure strategy equilibria maximize aggregate surplus. When matches are both supermodular and submodular, the allocation of buyers to sellers does not affect aggregate surplus,

as long as both buyers do not visit the same seller (which cannot occur in the pure strategy equilibria). However, in this case, pure strategy equilibria with both positive and negative assortative matching exist. This implies the existence of equilibria with mixing. At this point, it is useful to consider conditions under which this can occur.

Lemma 3: Match valuations are both supermodular and submodular in the following special cases:

a) Either buyers or sellers are homogeneous.
b) Heterogeneity is additive.

Proof: Straightforward.

In either of these two cases, equilibria exist where buyers mix in their location choice. These equilibria are clearly less efficient than the pure strategy equilibria, because they have an associated non-zero probability that both buyers will visit the same seller. However it can be shown that buyers choose the same probabilities that a planner would, if the planner were unable to avoid the coordination problem.

Proposition 2: Where equilibria exist with buyers mixing in their location choice these equilibria are constrained-efficient and the expected matching rate is strictly decreasing in the magnitude of the heterogeneity.

Proof: In the works.

CONCLUDING DISCUSSION

Heterogeneity can either worsen or improve coordination problems, depending on the form that it takes. If the cardinality of the types of agents and the number of agents is the same, and match valuations are either strictly supermodular or strictly submodular, then the existence of heterogeneity can eliminate inefficient equilibria with mixing. Otherwise, when mixing equilibria exist, then heterogeneity can worsen the coordination problem.

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