Supplement to “Unconditional Quantile Regressions” *

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Abstract

This supplement provides detailed derivation of the asymptotic properties of the estimators proposed in the article “Unconditional Quantile Regressions” by Firpo, Fortin and Lemieux.

Keywords: *Influence Functions, Unconditional Quantile, Quantile Regressions.*

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1 Large-sample properties of estimators of $UQPE(\tau)$

The first obvious aspect of the three estimation methods of $UQPE(\tau)$ is that all of them are based on using sample quantities $\widehat{f}_Y(\widehat{q}_\tau)$ and $\widehat{q}_\tau$. The density appears in the estimation of the parameter $c_{1,\tau}$ whereas the sample quantile is important by itself in the replacement of T_τ by $\widehat{T}_\tau$. Because the estimated density is used instead of the true one, our $UQPE(\tau)$ estimators will be nonparametric in essence, and therefore will converge at a slower rate than the usual parametric estimator. We will state the results regarding order of convergence and the asymptotic distribution of our estimators. Before that, we invoke the following assumptions that will be used throughout this appendix and that concern the distribution of Y.

Assumption 1 [***Conditions on the distribution of Y***] *(i)* $F_Y(\cdot)$ *is absolutely continuous and differentiable over* $y \in \mathbb{R}$ *and* $f_Y(y) = dF_Y(y)/dy$*; (ii)* $\int y \cdot f_Y(y) \cdot dy < \infty$*; (iii)* $f_Y(\cdot)$ *is uniformly continuous; (iv)* $\int |f_Y(y)| \cdot dy < \infty$*; (v)* $f_Y(\cdot)$ *is three times differentiable with bounded third derivative in a neighborhood of* y*; (vi) for* $\tau \in (0,1)$*, the sets* $\Upsilon_\tau = \inf_q \{F_Y(q) \geq \tau\}$*, are singletons and their elements* $q_\tau \in \Upsilon_\tau$ *satisfy* $q_\tau < \infty$ *and* $f_Y(q_\tau) > 0$.

We divide this section into several parts to facilitate the exposition. The first part concerns the rate of convergence and the limiting distribution of the components of $\widehat{\mathrm{RIF}}(y, \widehat{q}_\tau)$. Then we establish the results for the estimators of $UQPE(\tau)$.

1.1 Components of $\widehat{\mathrm{RIF}}(y, \widehat{q}_\tau)$

All estimators will involve $\widehat{f}_Y(\widehat{q}_\tau)$ and $\widehat{q}_\tau$ whenever $\widehat{c}_{1,\tau}$, $\widehat{T}$ and $\widehat{c}_{2,\tau}$ are used. We now establish some useful results regarding the limiting behavior of those quantities. Before that, let us suppose the following set of assumptions hold.

Assumption 2 [***Kernel Function and Bandwidth***] *(i)* $K_Y(\cdot)$ *is a bounded real-valued function satisfying (a)* $\int K_Y(y) \cdot dy = 1$*, (b)* $\int |K_Y(y)| \cdot dy < \infty$*, (c)* $\int K_Y^2(y) \cdot dy < \infty$*, (d)* $\lim_{y \to \pm\infty} |y| \cdot |K_Y(y)| = 0$*, (e)* $\sup_y |K_Y(y)| < \infty$*, (f)* $K_Y(y) = \frac{1}{2\pi} \int \exp(-i \cdot t \cdot y) \cdot \phi(t) \cdot dt$*, where* $\phi(t)$ *is the absolutely integrable characteristic function of* $K_Y(\cdot)$*; (ii)* b_Y *is a bandwidth sequence satisfying* $b_Y = O(N^{-a})$*,* $a < 1$*, and* $\lim_{N \to +\infty} \sqrt{N \cdot b_Y} \cdot b_Y^2 = 0$.

Now consider each individual component of $\widehat{\mathrm{RIF}}(y, \widehat{q}_\tau) = \widehat{c}_{1,\tau} \cdot \widehat{T}_\tau + \widehat{c}_{2,\tau}$:

1. $\widehat{c}_{1,\tau} - c_{1,\tau}$

   We have

$$
\begin{aligned}
\widehat{c}_{1,\tau} &= c_{1,\tau} + \frac{1}{\widehat{f}_Y\left(\widehat{q}_\tau\right)} - \frac{1}{f_Y\left(q_\tau\right)} \\
&= c_{1,\tau} + \frac{1}{\widehat{f}_Y\left(q_\tau\right)} - \frac{\widehat{f}'_Y\left(q_\tau\right)}{\widehat{f}^2_Y\left(q_\tau\right)} \cdot \left(\widehat{q}_\tau - q_\tau\right) - \frac{1}{f_Y\left(q_\tau\right)} + \eta_1,
\end{aligned}
$$

   where

$$
\eta_1 = -\left(\frac{\widehat{f}'_Y\left(\widetilde{q}_\tau\right)}{\widehat{f}^2_Y\left(\widetilde{q}_\tau\right)} - \frac{\widehat{f}'_Y\left(q_\tau\right)}{\widehat{f}^2_Y\left(q_\tau\right)}\right) \cdot \left(\widehat{q}_\tau - q_\tau\right) = O_p\left(\left|\widetilde{q}_\tau - q_\tau\right| \cdot \left|\widehat{q}_\tau - q_\tau\right|\right).
$$

   Note also that

$$
\frac{1}{\widehat{f}_Y\left(q_\tau\right)} = \frac{1}{f_Y\left(q_\tau\right)} - \frac{1}{f^2_Y\left(q_\tau\right)} \cdot \left(\widehat{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\right) + \eta_2,
$$

   with

$$
\begin{aligned}
\eta_2 &= -\left(\frac{1}{\widetilde{f}^2_Y\left(q_\tau\right)} - \frac{1}{f^2_Y\left(q_\tau\right)}\right) \cdot \left(\widehat{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\right) \\
&= O_p\Big(\Big|\widetilde{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\Big| \cdot \Big|\widehat{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\Big|\Big),
\end{aligned}
$$

   and, therefore,

$$
\frac{1}{\widehat{f}^2_Y\left(q_\tau\right)} = \frac{1}{f^2_Y\left(q_\tau\right)} - \frac{2}{f^3_Y\left(q_\tau\right)} \cdot \left(\widehat{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\right) + \eta'_2,
$$

   with

$$
\begin{aligned}
\eta'_2 &= -2 \cdot \left(\frac{1}{\widetilde{f}^3_Y\left(q_\tau\right)} - \frac{1}{f^3_Y\left(q_\tau\right)}\right) \cdot \left(\widehat{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\right) \\
&= O_p\left(\Big|\widetilde{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\Big| \cdot \Big|\widehat{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\Big|\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\widehat{c}_{1,\tau} &= c_{1,\tau} - \frac{1}{f_Y^2\left(q_\tau\right)} \cdot \left(\widehat{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\right) \\
&\quad - \widehat{f}'_Y\left(q_\tau\right) \cdot \left(\widehat{q}_\tau - q_\tau\right) \cdot \left(\frac{1}{f_Y^2\left(q_\tau\right)} - \frac{2}{f_Y^3\left(q_\tau\right)} \cdot \left(\widehat{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\right) + \eta'_2\right) + \eta_1 + \eta_2 \\
&= c_{1,\tau} - \frac{1}{f_Y\left(q_\tau\right)} \cdot \frac{\left(\widehat{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\right)}{f_Y\left(q_\tau\right)} \\
&\quad - \frac{1}{f_Y\left(q_\tau\right)} \cdot \left(1 - 2 \cdot \frac{\left(\widehat{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\right)}{f_Y\left(q_\tau\right)}\right) \cdot \frac{\widehat{f}'_Y\left(q_\tau\right)}{f_Y\left(q_\tau\right)} \cdot \left(\widehat{q}_\tau - q_\tau\right) + \eta_1 + \eta_2 + \eta_3,
\end{aligned}
$$

where

$$
\eta_3 = -\eta'_2 \cdot \widehat{f}'_Y\left(q_\tau\right) \cdot \left(\widehat{q}_\tau - q_\tau\right) = O_p\left(\left|\widehat{q}_\tau - q_\tau\right| \cdot \left|\widetilde{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\right| \cdot \left|\widehat{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right)\right|\right).
$$

Finally, because

$$
\begin{aligned}
\widehat{f}'_Y\left(q_\tau\right) &= f'_Y\left(q_\tau\right) + \widehat{f}'_Y\left(q_\tau\right) - f'_Y\left(q_\tau\right) \\
&= f'_Y\left(q_\tau\right) + \eta_4,
\end{aligned}
$$

with

$$
\eta_4 = \widehat{f}'_Y\left(q_\tau\right) - f'_Y\left(q_\tau\right) = O_p\left(\left|\widehat{f}'_Y\left(q_\tau\right) - f'_Y\left(q_\tau\right)\right|\right),
$$

we have that

$$
\begin{aligned}
\widehat{c}_{1,\tau} &= c_{1,\tau} - \frac{1}{f_Y(q_\tau)} \cdot \frac{\left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right)}{f_Y(q_\tau)} \\
&\quad - \frac{1}{f_Y(q_\tau)} \cdot \left(1 - 2 \cdot \frac{\left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right)}{f_Y(q_\tau)}\right) \cdot \frac{f'_Y(q_\tau)}{f_Y(q_\tau)} \cdot (\widehat{q}_\tau - q_\tau) \\
&\quad - \frac{1}{f_Y(q_\tau)} \cdot \left(1 - 2 \cdot \frac{\left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right)}{f_Y(q_\tau)}\right) \cdot \frac{(\widehat{q}_\tau - q_\tau)}{f_Y(q_\tau)} \cdot \eta_4 \\
&\quad + \eta_1 + \eta_2 + \eta_3 \\
&= c_{1,\tau} - \frac{1}{f_Y(q_\tau)} \cdot \frac{\left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right)}{f_Y(q_\tau)} \\
&\quad - \frac{1}{f_Y(q_\tau)} \cdot \left(1 - 2 \cdot \frac{\left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right)}{f_Y(q_\tau)}\right) \cdot \frac{f'_Y(q_\tau)}{f_Y(q_\tau)} \cdot (\widehat{q}_\tau - q_\tau) \\
&\quad + \eta_1 + \eta_2 + \eta_3 + \eta_5 \\
&= c_{1,\tau} - \frac{1}{f_Y(q_\tau)} \cdot \frac{\left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right)}{f_Y(q_\tau)} - \frac{1}{f_Y(q_\tau)} \cdot \frac{f'_Y(q_\tau)}{f_Y(q_\tau)} \cdot (\widehat{q}_\tau - q_\tau) + \eta,
\end{aligned}
$$

where

$$
\begin{aligned}
\eta_5 &= -\frac{1}{f_Y^2(q_\tau)} \cdot \left(1 - 2 \cdot \frac{\left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right)}{f_Y(q_\tau)}\right) \cdot \eta_4 \cdot (\widehat{q}_\tau - q_\tau) \\
&= O_p\left(\eta_4 \cdot |\widehat{q}_\tau - q_\tau| \cdot \left|\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right|\right) \\
&= O_p\left(\left|\widehat{f}'_Y(q_\tau) - f'_Y(q_\tau)\right| \cdot |\widehat{q}_\tau - q_\tau| \cdot \left|\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right|\right),
\end{aligned}
$$

$$
\begin{aligned}
\eta_6 &= 2 \cdot \frac{\left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right)}{f_Y^2(q_\tau)} \cdot \frac{f'_Y(q_\tau)}{f_Y(q_\tau)} \cdot (\widehat{q}_\tau - q_\tau) \\
&= O_p\left(|\widehat{q}_\tau - q_\tau| \cdot \left|\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right|\right),
\end{aligned}
$$

$$
\begin{aligned}
\eta &= \eta_1 + \eta_2 + \eta_3 + \eta_5 + \eta_6 \\
&= O_p\left(\left|\widetilde{q}_\tau - q_\tau\right| \cdot \left|\widehat{q}_\tau - q_\tau\right|\right) \\
&\quad + O_p\left(\left|\widetilde{f}_Y(q_\tau) - f_Y(q_\tau)\right| \cdot \left|\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right|\right) \\
&\quad + O_p\left(\left|\widehat{f}'_Y(q_\tau) - f'_Y(q_\tau)\right| \cdot \left|\widehat{q}_\tau - q_\tau\right| \cdot \left|\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right|\right).
\end{aligned}
$$

Now

$$
\begin{aligned}
\widehat{q}_\tau - q_\tau &= \frac{1}{N} \cdot \sum_{i=1}^{N} \frac{\tau - 1\!\!1\{Y_i \leq q_\tau\}}{f_Y(q_\tau)} + o_p\left(N^{-1/2}\right) \\
&= \frac{1}{N} \cdot \sum_{i=1}^{N} \frac{T_{\tau,i} - (1-\tau)}{f_Y(q_\tau)} + o_p\left(N^{-1/2}\right) \\
&= O_p\left(N^{-1/2}\right),
\end{aligned}
$$

while

$$
\begin{aligned}
\left|\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right| &= \left|\widehat{f}_Y(q_\tau) + E\left[\widehat{f}_Y(q_\tau)\right] - \left(E\left[\widehat{f}_Y(q_\tau)\right] - f_Y(q_\tau)\right)\right| \\
&\leq \left|\widehat{f}_Y(q_\tau) + E\left[\widehat{f}_Y(q_\tau)\right]\right| + \left|E\left[\widehat{f}_Y(q_\tau)\right] - f_Y(q_\tau)\right| \\
&= O_p\left((N \cdot b_Y)^{-1/2}\right) + O\left(b_Y^2\right).
\end{aligned}
$$

If we use a undersmoothing bandwidth that makes the bias vanishes faster than the variance (see Pagan and Ullah, 1999, for example), that is

$$
(N \cdot b_Y)^{1/2} \cdot b_Y^2 \underset{N\to\infty}{\longrightarrow} 0
$$

$$
b_Y = C \cdot N^{-a}\ , \qquad 1 > a > 1/5,
$$

then

$$
\begin{aligned}
\left|\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right| &= O_p\left(N^{(a-1)/2}\right) + O_p\left(N^{-2a}\right) \\
&= O_p\left(N^{(a-1)/2}\right),
\end{aligned}
$$

and, therefore

$$
(a-1)/2 > -2/5.
$$

Also suppose that

$$O_p\left(\left|\widehat{f}'_Y(q_\tau) - f'_Y(q_\tau)\right|\right) = O_p\left(N^{-b}\right) \ , \qquad b > 0.$$

Then

$$\eta = O_p\left(N^{\max\{a-1,-1,a/2-1-b\}}\right) = O_p\left(N^{a-1}\right),$$

because since $a > 1/5$ and $b > 0$

$$\begin{aligned} a-1 &> -1 \\ a-1 &> a/2-1-b \Leftarrow a+2b>0 \Leftarrow a>1/5 \text{ and } b>0. \end{aligned}$$

Now, we have

$$\begin{aligned} \widehat{c}_{1,\tau} &= c_{1,\tau} - \frac{1}{f_Y(q_\tau)} \cdot \frac{\left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right)}{f_Y(q_\tau)} - \frac{1}{f_Y(q_\tau)} \cdot \frac{f'_Y(q_\tau)}{f_Y(q_\tau)} \cdot (\widehat{q}_\tau - q_\tau) + \eta \\ &= c_{1,\tau} + O_p\left(N^{(a-1)/2}\right) + O_p\left(N^{-1/2}\right) + O_p\left(N^{a-1}\right). \end{aligned}$$

Finally

$$\begin{aligned} \widehat{c}_{1,\tau} - c_{1,\tau} &= -\frac{1}{f_Y(q_\tau)} \cdot \frac{\left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right)}{f_Y(q_\tau)} + o_p\left(N^{(a-1)/2}\right) \\ &= -\frac{1}{f_Y(q_\tau)} \cdot \frac{\left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right)}{f_Y(q_\tau)} + o_p\left((N \cdot b_Y)^{-1/2}\right), \end{aligned}$$

and

$$\begin{aligned} \sqrt{N \cdot b_Y} \cdot (\widehat{c}_{1,\tau} - c_{1,\tau}) &= -\frac{1}{f_Y^2(q_\tau)} \cdot \left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right) + o_p(1) \\ &\xrightarrow{D} N\left(0, \frac{1}{f_Y^3(q_\tau)} \cdot \int K_Y^2(z) \cdot dz\right). \end{aligned}$$

2. $\widehat{c}_{2,\tau} - c_{2,\tau}$

We have

$$
\begin{aligned}
\widehat{c}_{2,\tau} &= c_{2,\tau} + \widehat{q}_\tau + \widehat{c}_{1,\tau} \cdot (\tau - 1) - (q_\tau + c_{1,\tau} \cdot (\tau - 1)) \\
&= c_{2,\tau} + \widehat{q}_\tau - q_\tau + (\tau - 1) \cdot (\widehat{c}_{1,\tau} - c_{1,\tau}) \\
&= c_{2,\tau} + O_p\left(N^{-1/2}\right) + O_p\left(N^{(a-1)/2}\right) \\
&= c_{2,\tau} + O_p\left(N^{(a-1)/2}\right).
\end{aligned}
$$

Thus

$$
\widehat{c}_{2,\tau} - c_{2,\tau} = (\tau - 1) \cdot (\widehat{c}_{1,\tau} - c_{1,\tau}) + o_p\left(N^{(a-1)/2}\right),
$$

and

$$
\begin{aligned}
\sqrt{N \cdot b_Y} \cdot (\widehat{c}_{2,\tau} - c_{2,\tau}) &= \frac{(1-\tau)}{f_Y^2(q_\tau)} \cdot \left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right) + o_p(1) \\
&\xrightarrow{D} N\left(0, \frac{(1-\tau)^2}{f_Y^3(q_\tau)} \cdot \int K_Y^2(z) \cdot dz\right).
\end{aligned}
$$

3. $\frac{1}{N} \cdot \sum_{i=1}^{N} \left(\widehat{T}_{\tau,i} - T_{\tau,i}\right)$

Define

$$
R(Y_i, \widehat{q}_\tau, q_\tau) \equiv \widehat{T}_{\tau,i} - T_{\tau,i}.
$$

Then

$$
\begin{aligned}
&\left|\frac{1}{\sqrt{N}} \cdot \sum_{i=1}^{N} \left(R(Y_i, \widehat{q}_\tau, q_\tau) - E\left[R(Y, \widehat{q}_\tau, q_\tau)\right]\right)\right| \\
\leq\ & O_p\left(\left(E\left[E\left[R^2(Y, \widehat{q}_\tau, q_\tau) \,|\widehat{q}_\tau\right]\right]\right)^{1/2}\right) \\
=\ & O_p\left(\left(E\left[\Pr\left[q_\tau < Y \leq \widehat{q}_\tau | \widehat{q}_\tau\right]\right]\right)^{1/2}\right) \\
=\ & O_p\left(\left(E\left[\sup_{\widehat{q}_\tau \in \mathbb{R}} f_{Y|\widehat{q}_\tau}(\widetilde{q}_\tau | \widehat{q}_\tau) \,|\widehat{q}_\tau\right] \cdot |\widehat{q}_\tau - q_\tau|\right)^{1/2}\right) \\
=\ & O_p\left(|\widehat{q}_\tau - q_\tau|^{1/2}\right) \\
=\ & O_p\left(N^{-1/4}\right),
\end{aligned}
$$

so

$$\begin{aligned}
& \left| \frac{1}{N} \cdot \sum_{i=1}^{N} R\left(Y_i, \widehat{q}_\tau, q_\tau\right) \right| \\
\leq\ & \left| E\left[R\left(Y, \widehat{q}_\tau, q_\tau\right)\right] \right| + N^{-1/2} \cdot \left| \frac{1}{\sqrt{N}} \cdot \sum_{i=1}^{N} \left(R\left(Y_i, \widehat{q}_\tau, q_\tau\right) - E\left[R\left(Y, \widehat{q}_\tau, q_\tau\right)\right]\right) \right| \\
=\ & O_p\left(\left|E\left[R^2\left(Y, \widehat{q}_\tau, q_\tau\right)\right]\right|\right) + O_p\left(N^{-3/4}\right) \\
=\ & O_p\left(\left|\widehat{q}_\tau - q_\tau\right|\right) + O_p\left(N^{-3/4}\right) = O_p\left(N^{-1/2}\right) + O_p\left(N^{-3/4}\right) \\
=\ & O_p\left(N^{-1/2}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{N} \cdot \sum_{i=1}^{N} \left(\widehat{T}_{\tau,i} - T_{\tau,i}\right) &= \widehat{q}_\tau - q_\tau + o_p\left(N^{-1/2}\right) \\
&= \frac{1}{N} \cdot \sum_{i=1}^{N} \frac{T_{\tau,i} - (1-\tau)}{f_Y\left(q_\tau\right)} + o_p\left(N^{-1/2}\right) \\
&\overset{D}{\rightarrow} N\left(0, \frac{\tau \cdot (1-\tau)}{f_Y^2\left(q_\tau\right)}\right).
\end{aligned}$$

1.2 RIF $- OLS$: asymptotics

Specific assumptions for the particular estimators RIF $- OLS$ are:

Assumption 3 [RIF $-$ ***OLS***] *(i)* $E\left[X \cdot X^\intercal\right]$ *is invertible and (ii)* $Cov\left[X \cdot \mathbb{1}\left\{Y > q_\tau\right\}\right] \neq 0$*, (iii)* $E\left[vec\left(X \cdot X^\intercal\right) \cdot vec\left(X \cdot X^\intercal\right)^\intercal\right] < \infty$*,* $E\left[X \cdot X^\intercal \cdot \mathbb{1}\left\{Y > q_\tau\right\}\right] < \infty$*; (iv)* $c_{1,\tau} \cdot E\left[\mathbb{1}\left\{Y > q_\tau\right\} | X = x\right] + c_{2,\tau} = x^\intercal \gamma_\tau$.

The key result of this subsection is:

Proposition 1 [RIF $-$ ***OLS***] *Under assumptions* ***1, 2 and 3***

$$\sqrt{N \cdot b_Y} \cdot \left(\widehat{\gamma}_\tau - \gamma_\tau\right) \overset{D}{\rightarrow}$$
$$N\left(0, \left(E\left[X \cdot X^\intercal\right]\right)^{-1} Cov\left[X, T_\tau\right] \cdot Cov\left[X, T_\tau\right]^\intercal \cdot \left(E\left[X \cdot X^\intercal\right]\right)^{-1} \cdot f_Y^{-3}\left(q_\tau\right) \cdot \int K_Y^2(z) \cdot dz\right)$$

Proof of Proposition 1: We divide the proof into two parts. We first find the convergence rate of the estimator and then derive its asymptotic distribution.

1.2.1 Order of convergence

The difference between the estimator and the population parameter is

$$\widehat{\gamma}_\tau - \gamma_\tau = \widehat{\gamma}_\tau - \widetilde{\gamma}_\tau + \widetilde{\gamma}_\tau - \gamma_\tau$$

where we define $\widetilde{\gamma}_\tau$ as the coefficient that we would compute if we were using the true RIF:

$$\widetilde{\gamma}_\tau = \left(\sum_{i=1}^{N} X_i \cdot X_i^\intercal \right)^{-1} \cdot \sum_{i=1}^{N} X_i \cdot \mathrm{RIF}(Y_i; q_\tau),$$

and γ_τ is

$$\gamma_\tau = \left(E\left[X \cdot X^\intercal\right] \right)^{-1} \cdot E\left[X \cdot \mathrm{RIF}(Y; q_\tau)\right].$$

The first term $\widehat{\gamma}_\tau - \widetilde{\gamma}_\tau$ can be obtained by rewriting $\widehat{\gamma}_\tau$ as

$$\begin{aligned}
\widehat{\gamma}_\tau &= \left(\sum_{i=1}^{N} X_i \cdot X_i^\intercal \right)^{-1} \cdot \sum_{i=1}^{N} X_i \cdot \widehat{\mathrm{RIF}}(Y_i; \widehat{q}_\tau) \\
&= \left(\sum_{i=1}^{N} X_i \cdot X_i^\intercal \right)^{-1} \cdot \sum_{i=1}^{N} X_i \cdot \mathrm{RIF}(Y_i; q_\tau) \\
&\quad + \left(\sum_{i=1}^{N} X_i \cdot X_i^\intercal \right)^{-1} \cdot \sum_{i=1}^{N} X_i \cdot \left(\widehat{\mathrm{RIF}}(Y_i; \widehat{q}_\tau) - \mathrm{RIF}(Y_i; q_\tau) \right) \\
&= \widetilde{\gamma}_\tau + \left(\sum_{i=1}^{N} X_i \cdot X_i^\intercal \right)^{-1} \cdot \sum_{i=1}^{N} X_i \cdot \left((\widehat{c}_{1,\tau} - c_{1,\tau}) \cdot \widehat{T}_{\tau,i} + c_{1,\tau} \cdot \left(\widehat{T}_{\tau,i} - T_{\tau,i} \right) + \widehat{c}_{2,\tau} - c_{2,\tau} \right) \\
&= \widetilde{\gamma}_\tau + \left(\frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot X_i^\intercal \right)^{-1} \\
&\quad \cdot \sum_{i=1}^{N} X_i \cdot \left((\widehat{c}_{1,\tau} - c_{1,\tau}) \cdot \left(\widehat{T}_{\tau,i} - (1-\tau) \right) + c_{1,\tau} \cdot \left(\widehat{T}_{\tau,i} - T_{\tau,i} \right) + \widehat{q}_\tau - q_\tau \right)
\end{aligned}$$

therefore

$$
\begin{aligned}
& \left\|\widehat{\gamma}_\tau-\widetilde{\gamma}_\tau\right\| \\
= & \left(\frac{1}{N} \cdot \sum_{i=1}^N X_i \cdot X_i^{\intercal}\right)^{-1} \cdot\left(\left(\widehat{c}_{1, \tau}-c_{1, \tau}\right) \cdot \frac{1}{N} \cdot \sum_{i=1}^N X_i \cdot\left(\widehat{T}_{\tau, i}-(1-\tau)\right)\right. \\
& \left.\quad+\frac{c_{1, \tau}}{N} \cdot \sum_{i=1}^N X_i \cdot\left(\widehat{T}_{\tau, i}-T_{\tau, i}\right)+\left(\widehat{q}_\tau-q_\tau\right) \cdot \frac{1}{N} \cdot \sum_{i=1}^N X_i\right) \\
\leq & C \cdot\left|\widehat{c}_{1, \tau}-c_{1, \tau}\right| \cdot\left\|\frac{1}{N} \cdot \sum_{i=1}^N X_i \cdot\left(T_{\tau, i}-(1-\tau)\right)\right\| \\
& +C \cdot\left|\widehat{c}_{1, \tau}-c_{1, \tau}\right| \cdot\left\|\frac{1}{N} \cdot \sum_{i=1}^N X_i \cdot\left(\widehat{T}_{\tau, i}-T_{\tau, i}\right)\right\| \\
& +C \cdot\left\|\frac{c_{1, \tau}}{N} \cdot \sum_{i=1}^N X_i \cdot\left(\widehat{T}_{\tau, i}-T_{\tau, i}\right)\right\|+C \cdot\left|\widehat{q}_\tau-q_\tau\right| \cdot\left\|\frac{1}{N} \cdot \sum_{i=1}^N X_i\right\|,
\end{aligned}
$$

but

$$
\begin{aligned}
& \left\|\frac{1}{\sqrt{N}} \cdot\left(\sum_{i=1}^N X_i \cdot\left(\widehat{T}_{\tau, i}-T_{\tau, i}\right)-E\left[X \cdot\left(\widehat{T}_\tau-T_\tau\right)\right]\right)\right\| \\
= & \left\|\frac{1}{\sqrt{N}} \cdot \sum_{i=1}^N\left(X_i \cdot R\left(Y_i, \widehat{q}_\tau, q_\tau\right)-E\left[X \cdot R\left(Y, \widehat{q}_\tau, q_\tau\right)\right]\right)\right\| \\
\leq & O_p\left(\left\|E\left[X \cdot X^{\intercal} \cdot E\left[E\left[R^2\left(Y, \widehat{q}_\tau, q_\tau\right) \mid \widehat{q}_\tau, X\right] \mid X\right]\right]\right\|^{1 / 2}\right) \\
= & O_p\left(\left\|E\left[X \cdot X^{\intercal} \cdot E\left[\operatorname{Pr}\left[q_\tau<Y \leq \widehat{q}_\tau \mid \widehat{q}_\tau, X\right] X\right]\right]\right\|^{1 / 2}\right) \\
= & O_p\left(\left\|E\left[E\left[X \cdot X^{\intercal} \cdot E\left[\sup _{\widehat{q}_\tau \in \mathbb{R}} f_{Y \mid \widehat{q}_\tau, X}\left(\widetilde{q}_\tau \mid \widehat{q}_\tau, X\right) \mid \widehat{q}_\tau, X\right] \mid X\right]\right] \cdot\left|\widehat{q}_\tau-q_\tau\right|\right\|^{1 / 2}\right) \\
= & \left\|E\left[X \cdot X^{\intercal}\right]\right\|^{1 / 2} \cdot O_p\left(\left|\widehat{q}_\tau-q_\tau\right|^{1 / 2}\right) \\
= & O_p\left(\left|\widehat{q}_\tau-q_\tau\right|^{1 / 2}\right) \\
= & O_p\left(N^{-1 / 4}\right),
\end{aligned}
$$

and, therefore

$$
\begin{aligned}
&\left\| \frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot \left(\widehat{T}_{\tau,i} - T_{\tau,i} \right) \right\| \\
\leq\ & \left\| E\left[X \cdot \left(\widehat{T}_{\tau} - T_{\tau} \right) \right] \right\| + N^{-1/2} \cdot \left\| \frac{1}{\sqrt{N}} \cdot \left(\sum_{i=1}^{N} X_i \cdot \left(\widehat{T}_{\tau,i} - T_{\tau,i} \right) - E\left[X \cdot \left(\widehat{T}_{\tau} - T_{\tau} \right) \right] \right) \right\| \\
=\ & O_p\left(\left| E\left[X \cdot R^2\left(Y, \widehat{q}_{\tau}, q_{\tau} \right) \right] \right| \right) + O_p\left(N^{-3/4} \right) \\
=\ & \| E[X] \| \cdot O_p\left(\left| \widehat{q}_{\tau} - q_{\tau} \right| \right) + O_p\left(N^{-3/4} \right) = O_p(1) \cdot O_p\left(N^{-1/2} \right) + O_p\left(N^{-3/4} \right) \\
=\ & O_p\left(N^{-1/2} \right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\| \widehat{\gamma}_{\tau} - \widetilde{\gamma}_{\tau} \| &= O_p\left(\left| \widehat{c}_{1,\tau} - c_{1,\tau} \right| \right) \cdot O_p(1) + O_p\left(\left| \widehat{c}_{1,\tau} - c_{1,\tau} \right| \right) \cdot O_p\left(N^{-1/2} \right) + O_p\left(N^{-1/2} \right) \\
&= O_p\left(N^{(a-1)/2} \right) + O_p\left(N^{-1/2} \right).
\end{aligned}
$$

Since $a > 1/5$, it follows that

$$
\left| \widehat{\gamma}_{\tau} - \widetilde{\gamma}_{\tau} \right| = O_p\left(N^{(a-1)/2} \right).
$$

Now we look at the second term,

$$
\begin{aligned}
&\| \widetilde{\gamma}_{\tau} - \gamma_{\tau} \| = \left\| \left(\frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot X_i^{\intercal} \right)^{-1} \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot \mathrm{RIF}(Y_i; q_{\tau}) - \gamma_{\tau} \right\| \\
\leq\ & \left\| \left(\left(\frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot X_i^{\intercal} \right)^{-1} - \left(E\left[X \cdot X^{\intercal} \right] \right)^{-1} \right) \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot \mathrm{RIF}(Y_i; q_{\tau}) \right\| \\
& + \left\| \left(E\left[X \cdot X^{\intercal} \right] \right)^{-1} \cdot \left(\frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot \mathrm{RIF}(Y_i; q_{\tau}) - E\left[X \cdot \mathrm{RIF}(Y; q_{\tau}) \right] \right) \right\| \\
\leq\ & \frac{1}{\sqrt{N}} \cdot \left\| \sqrt{N} \cdot \left(\left(\frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot X_i^{\intercal} \right)^{-1} - \left(E\left[X \cdot X^{\intercal} \right] \right)^{-1} \right) \right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot \mathrm{RIF}(Y_i; q_{\tau}) \right\| \\
& + \frac{1}{\sqrt{N}} \cdot \left\| \left(E\left[X \cdot X^{\intercal} \right] \right)^{-1} \right\| \cdot \left\| \sqrt{N} \cdot \left(\frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot \mathrm{RIF}(Y_i; q_{\tau}) - E\left[X \cdot \mathrm{RIF}(Y; q_{\tau}) \right] \right) \right\| \\
=\ & N^{-1/2} \cdot O_p(1) \cdot O_p(1) + N^{-1/2} \cdot O_p(1) \cdot O_p(1) \\
=\ & O_p\left(N^{-1/2} \right).
\end{aligned}
$$

Therefore
$$\widehat{\gamma}_\tau - \gamma_\tau = O_p\left(N^{(a-1)/2}\right),$$
that is, the difference between the feasible and the unfeasible estimators determines the rate of convergence.

1.2.2 Asymptotic Distribution

$$\widehat{\gamma}_\tau - \gamma_\tau = \left(\widehat{c}_{1,\tau} - c_{1,\tau}\right) \cdot \left(\frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot X_i^{\intercal}\right)^{-1} \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot \left(T_{\tau,i} - (1-\tau)\right) + o_p\left(N^{(a-1)/2}\right),$$
so we have that
$$\begin{aligned}
& \sqrt{N \cdot b_Y} \cdot \left(\widehat{\gamma}_\tau - \gamma_\tau\right) \\
= \; & \left(\widehat{c}_{1,\tau} - c_{1,\tau}\right) \cdot \left(E\left[X \cdot X^{\intercal}\right]\right)^{-1} \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot \left(T_{\tau,i} - (1-\tau)\right) + o_p(1) \\
= \; & -\frac{1}{f_Y^2(q_\tau)} \cdot \left(E\left[X \cdot X^{\intercal}\right]\right)^{-1} \cdot Cov\left[X, T_\tau\right] \cdot \sqrt{N \cdot b_Y} \cdot \left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right) \\
& -\frac{\left(E\left[X \cdot X^{\intercal}\right]\right)^{-1}}{f_Y^2(q_\tau)} \cdot \sqrt{N \cdot b_Y} \cdot \left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right) \\
& \cdot \left(\frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot \left(T_{\tau,i} - (1-\tau)\right) - Cov\left[X, T_\tau\right]\right) + o_p(1) \\
= \; & -\frac{1}{f_Y^2(q_\tau)} \cdot \left(E\left[X \cdot X^{\intercal}\right]\right)^{-1} \cdot Cov\left[X, T_\tau\right] \cdot \sqrt{N \cdot b_Y} \cdot \left(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)\right) + o_p(1) \\
& \xrightarrow{D} N\left(0, V_{OLS}\right),
\end{aligned}$$
where
$$\begin{aligned}
V_{OLS} = & \; f_Y^{-3}(q_\tau) \cdot \left(E\left[X \cdot X^{\intercal}\right]\right)^{-1} \cdot Cov\left[X, T_\tau\right] \cdot Cov\left[X, T_\tau\right]^{\intercal} \cdot \left(E\left[X \cdot X^{\intercal}\right]\right)^{-1} \cdot \int K_Y^2(z) \cdot dz \\
= & \left(\gamma_\tau - (1-\tau) \cdot \left(E\left[X \cdot X^{\intercal}\right]\right)^{-1} \cdot E[X]\right) \cdot \left(\gamma_\tau - (1-\tau) \cdot \left(E\left[X \cdot X^{\intercal}\right]\right)^{-1} \cdot E[X]\right)^{\intercal} \\
& \cdot f_Y^{-3}(q_\tau) \cdot \int K_Y^2(z) \cdot dz,
\end{aligned}$$

and

$$
\begin{aligned}
Cov\left[X, T_{\tau}\right] &= E\left[\mathbb{1}\left\{Y > q_{\tau}\right\} \cdot X\right] - (1-\tau) \cdot E\left[X\right] \\
m_{\tau}\left(x\right) &= E\left[E\left[\mathbb{1}\left\{Y > q_{\tau}\right\} \left|X\right] \cdot X\right] - (1-\tau) \cdot E\left[X\right] \\
&= E\left[\Pr\left[Y > q_{\tau} \middle| X\right] \cdot X\right] - \Pr\left[Y > q_{\tau}\right] \cdot E\left[X\right] \\
&= E\left[m_{\tau}\left(X\right) \cdot X\right] - (1-\tau) \cdot E\left[X\right] \\
&= E\left[X \cdot X^{\intercal}\right] \cdot \gamma_{\tau} - (1-\tau) E\left[X\right].
\end{aligned}
$$

■

1.3 RIF $-$ *logit*: asymptotics

We make the following assumptions:

Assumption 4 [RIF $-$ ***Logit***] *(i)* $E\left[\left(\Lambda''\left(X^{\intercal}\theta_{\tau}^{*}\right)\right)^{2} \cdot X \cdot X^{\intercal}\right]$ *is invertible for all* $\theta_{\tau}^{*} \in \mathbb{R}^{k}$, *(ii)*$E\left[\left(1 - \Lambda\left(X^{\intercal}\theta_{\tau}\right)\right)^{2} \cdot \Lambda^{2}\left(X^{\intercal}\theta_{\tau}\right)\right] < \infty$, *(iii)* $E\left[\mathbb{1}\left\{Y > q_{\tau}\right\} \left| X = x\right.\right] = \Lambda\left(x^{\intercal}\theta_{\tau}\right)$.

The key result of this subsection is:

Proposition 2 [RIF $-$ ***Logit***] *Under assumptions* ***1, 2 and 4***

$$
\begin{aligned}
&\sqrt{N \cdot b_{Y}} \cdot \left(\widehat{UQPE}_{\text{RIF}-logit}\left(\tau\right) - UQPE\left(\tau\right)\right) \xrightarrow{D} \\
&\qquad N\left(0, f_{Y}^{-3}\left(q_{\tau}\right) \cdot E\left[\Lambda'\left(X^{\intercal}\theta_{\tau}\right)\right]^{2} \cdot \theta_{\tau} \cdot \theta_{\tau}^{\intercal} \cdot \int K_{Y}^{2}\left(z\right) \cdot dz\right)
\end{aligned}
$$

Proof of Proposition 2: We divide the proof into two parts. We first find the convergence rate of the estimator and then derive its asymptotic distribution.

1.3.1 Order of convergence

We estimate $UQPE(\tau)$ under the logistic assumption as $\widehat{UQPE}_{\mathrm{RIF}-logit}(\tau)$, where[1]

$$\widehat{UQPE}_{\mathrm{RIF}-logit}(\tau) = \widehat{c}_{1,\tau} \cdot \widehat{\theta}_{\tau} \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^{\intercal}\widehat{\theta}_{\tau}\right).$$

An unfeasible estimator of $UQPE(\tau)$ uses the unobservable variable T_{τ} instead of $\widehat{T}_{\tau}$ and the true constant $c_{1,\tau}$ instead of $\widehat{c}_{1,\tau}$. We call the unfeasible estimator $\widetilde{UQPE}_{\mathrm{RIF}-logit}(\tau)$:

$$\widetilde{UQPE}_{\mathrm{RIF}-logit}(\tau) = c_{1,\tau} \cdot \widetilde{\theta}_{\tau} \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^{\intercal}\widetilde{\theta}_{\tau}\right),$$

where

$$\widetilde{\theta}_{\tau} = \arg\max_{\theta_{\tau}} \sum_{i}^{N} T_{\tau,i} \cdot X_i^{\intercal}\theta_{\tau} + \log\left(1 - \Lambda\left(X_i^{\intercal}\theta_{\tau}\right)\right).$$

We now find the probability order of $\widehat{UQPE}_{\mathrm{RIF}-logit} - UQPE(\tau)$:

$$\begin{aligned} & \left\|\widehat{UQPE}_{\mathrm{RIF}-logit} - UQPE(\tau)\right\| \\ \leq & \left\|\widehat{UQPE}_{\mathrm{RIF}-logit} - \widetilde{UQPE}_{\mathrm{RIF}-logit}(\tau)\right\| + \left\|\widetilde{UQPE}_{\mathrm{RIF}-logit}(\tau) - UQPE(\tau)\right\|, \end{aligned}$$

where

[1] Remember that by the properties of the logistic distribution:

$$\begin{aligned} \Lambda'(z) &= \Lambda(z) \cdot (1 - \Lambda(z)) \\ \Lambda''(z) &= \Lambda'(z) \cdot (1 - 2 \cdot \Lambda(z)) \end{aligned}$$

$$
\begin{aligned}
&\left\|\widehat{UQPE}_{\mathrm{RIF}-logit}-\widetilde{UQPE}_{\mathrm{RIF}-logit}(\tau)\right\| \\
= &\left\|\widehat{c}_{1,\tau}\cdot\widehat{\theta}_\tau\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)-c_{1,\tau}\cdot\widetilde{\theta}_\tau\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\Lambda'\left(X_i^\intercal\widetilde{\theta}_\tau\right)\right\| \\
\leq &\left\|\left(\widehat{c}_{1,\tau}-c_{1,\tau}\right)\cdot\widehat{\theta}_\tau\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)\right\| \\
&+c_{1,\tau}\cdot\left\|\widehat{\theta}_\tau\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)-\widetilde{\theta}_\tau\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\Lambda'\left(X_i^\intercal\widetilde{\theta}_\tau\right)\right\| \\
\leq &\left\|\left(\widehat{c}_{1,\tau}-c_{1,\tau}\right)\cdot\widetilde{\theta}_\tau\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)\right\|+\left\|\left(\widehat{c}_{1,\tau}-c_{1,\tau}\right)\cdot\left(\widehat{\theta}_\tau-\widetilde{\theta}_\tau\right)\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)\right\| \\
&+c_{1,\tau}\cdot\left\|\left(\widehat{\theta}_\tau-\widetilde{\theta}_\tau\right)\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)\right\|+c_{1,\tau}\cdot\left\|\widetilde{\theta}_\tau\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\left(\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)-\Lambda'\left(X_i^\intercal\widetilde{\theta}_\tau\right)\right)\right\|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left\|\widehat{UQPE}_{\mathrm{RIF}-logit}-\widetilde{UQPE}_{\mathrm{RIF}-logit}(\tau)\right\| \\
\leq &\left\|\left(\widehat{c}_{1,\tau}-c_{1,\tau}\right)\cdot\theta_\tau\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)\right\| \\
&+\left\|\left(\widehat{c}_{1,\tau}-c_{1,\tau}\right)\cdot\left(\widetilde{\theta}_\tau-\theta_\tau\right)\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)\right\| \\
&+\left\|\left(\widehat{c}_{1,\tau}-c_{1,\tau}\right)\cdot\left(\widehat{\theta}_\tau-\widetilde{\theta}_\tau\right)\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)\right\| \\
&+c_{1,\tau}\cdot\left\|\left(\widehat{\theta}_\tau-\widetilde{\theta}_\tau\right)\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)\right\| \\
&+c_{1,\tau}\cdot\left\|\theta_\tau\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\left(\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)-\Lambda'\left(X_i^\intercal\widetilde{\theta}_\tau\right)\right)\right\| \\
&+c_{1,\tau}\cdot\left\|\left(\widetilde{\theta}_\tau-\theta_\tau\right)\cdot\frac{1}{N}\cdot\sum_{i=1}^{N}\left(\Lambda'\left(X_i^\intercal\widehat{\theta}_\tau\right)-\Lambda'\left(X_i^\intercal\widetilde{\theta}_\tau\right)\right)\right\|,
\end{aligned}
$$

and, therefore

$$\begin{aligned}
& \left\| \widehat{UQPE}_{\text{RIF}-logit} - \widetilde{UQPE}_{\text{RIF}-logit}(\tau) \right\| \\
\leq \; & |\widehat{c}_{1,\tau} - c_{1,\tau}| \cdot \|\theta_\tau\| \cdot \left| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right) \right| \\
& + |\widehat{c}_{1,\tau} - c_{1,\tau}| \cdot \|\theta_\tau\| \cdot \left\| \widehat{\theta}_\tau - \widetilde{\theta}_\tau \right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal \right\| \\
& + |\widehat{c}_{1,\tau} - c_{1,\tau}| \cdot \left\| \widetilde{\theta}_\tau - \theta_\tau \right\| \cdot \left| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right) \right| \\
& + |\widehat{c}_{1,\tau} - c_{1,\tau}| \cdot \left\| \widehat{\theta}_\tau - \widetilde{\theta}_\tau \right\| \cdot \left\| \widetilde{\theta}_\tau - \theta_\tau \right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal \right\| \\
& + |\widehat{c}_{1,\tau} - c_{1,\tau}| \cdot \left\| \widehat{\theta}_\tau - \widetilde{\theta}_\tau \right\| \cdot \left| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right) \right| \\
& + |\widehat{c}_{1,\tau} - c_{1,\tau}| \cdot \left\| \widehat{\theta}_\tau - \widetilde{\theta}_\tau \right\|^2 \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal \right\| \\
& + c_{1,\tau} \cdot \left\| \widehat{\theta}_\tau - \widetilde{\theta}_\tau \right\| \cdot \left| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right) \right| + c_{1,\tau} \cdot \left\| \widehat{\theta}_\tau - \widetilde{\theta}_\tau \right\|^2 \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal \right\| \\
& + c_{1,\tau} \cdot \|\theta_\tau\| \cdot \left\| \widehat{\theta}_\tau - \widetilde{\theta}_\tau \right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal \right\| \\
& + c_{1,\tau} \cdot \left\| \widehat{\theta}_\tau - \widetilde{\theta}_\tau \right\| \cdot \left\| \widetilde{\theta}_\tau - \theta_\tau \right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal \right\|,
\end{aligned}$$

where θ_τ^* is an intermediate value between $\widehat{\theta}_\tau$ and $\widetilde{\theta}_\tau$. We now give names to the terms above and find their convergence rates:

$$\varphi_1 = |\widehat{c}_{1,\tau} - c_{1,\tau}| \cdot \|\theta_\tau\| \cdot \left| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right) \right|,$$

$$\varphi_2 = |\widehat{c}_{1,\tau} - c_{1,\tau}| \cdot \|\theta_\tau\| \cdot \left\| \widehat{\theta}_\tau - \widetilde{\theta}_\tau \right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal \right\|,$$

$$\varphi_3 = |\widehat{c}_{1,\tau} - c_{1,\tau}| \cdot \left\| \widetilde{\theta}_\tau - \theta_\tau \right\| \cdot \left| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right) \right|,$$

$$\varphi_4 = |\widehat{c}_{1,\tau} - c_{1,\tau}| \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\| \cdot \left\|\widetilde{\theta}_\tau - \theta_\tau\right\| \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal\right\|,$$

$$\varphi_5 = |\widehat{c}_{1,\tau} - c_{1,\tau}| \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\| \cdot \left|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right)\right|,$$

$$\varphi_6 = |\widehat{c}_{1,\tau} - c_{1,\tau}| \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\|^2 \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal\right\|,$$

$$\varphi_7 = c_{1,\tau} \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\| \cdot \left|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right)\right|,$$

$$\varphi_8 = c_{1,\tau} \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\|^2 \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal\right\|,$$

$$\varphi_9 = c_{1,\tau} \cdot \|\theta_\tau\| \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\| \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal\right\|,$$

$$\varphi_{10} = c_{1,\tau} \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\| \cdot \left\|\widetilde{\theta}_\tau - \theta_\tau\right\| \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal\right\|.$$

Now we find a bound for the difference $\widehat{\theta}_\tau - \widetilde{\theta}_\tau$. Remember first that the first order conditions imply that $\widehat{\theta}_\tau$ and $\widetilde{\theta}_\tau$ are solutions to:

$$\begin{aligned}
\sum_{i=1}^{N} X_i^\intercal \cdot \left(\widehat{T}_{\tau,i} - \Lambda\left(X_i^\intercal \widehat{\theta}_\tau\right)\right) &= 0 \\
\sum_{i=1}^{N} X_i^\intercal \cdot \left(T_{\tau,i} - \Lambda\left(X_i^\intercal \widetilde{\theta}_\tau\right)\right) &= 0.
\end{aligned}$$

Thus,

$$
\begin{aligned}
& \sum_{i=1}^{N} X_i \cdot \left(\widehat{T}_{\tau,i} - \Lambda\left(X_i^{\intercal}\widehat{\theta}_\tau\right)\right) \\
= & \sum_{i=1}^{N} X_i \cdot \left(T_{\tau,i} - \Lambda\left(X_i^{\intercal}\widehat{\theta}_\tau\right) + \left(\widehat{T}_{\tau,i} - T_{\tau,i}\right)\right) \\
= & \sum_{i=1}^{N} X_i \cdot \left(T_{\tau,i} - \Lambda\left(X_i^{\intercal}\widetilde{\theta}_\tau\right) - X_i^{\intercal} \cdot \Lambda'\left(X_i^{\intercal}\theta_\tau^*\right) \cdot \left(\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right) + \left(\widehat{T}_{\tau,i} - T_{\tau,i}\right)\right) \\
= & -\sum_{i=1}^{N} X_i \cdot \left(X_i^{\intercal} \cdot \Lambda'\left(X_i^{\intercal}\theta_\tau^*\right) \cdot \left(\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right) - \left(\widehat{T}_{\tau,i} - T_{\tau,i}\right)\right) = 0
\end{aligned}
$$

The difference between the two solutions $\widehat{\theta}_\tau$ and $\widetilde{\theta}_\tau$ is:

$$
\begin{aligned}
& \widehat{\theta}_\tau - \widetilde{\theta}_\tau \\
= & \left(\frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot X_i^{\intercal} \cdot \Lambda'\left(X_i^{\intercal}\theta_\tau^*\right)\right)^{-1} \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot \left(\widehat{T}_{\tau,i} - T_{\tau,i}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\| & \leq C \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N} X_i \cdot \left(\widehat{T}_{\tau,i} - T_{\tau,i}\right)\right\| \\
& = O_p\left(\left|\widehat{q}_\tau - q_\tau\right|\right) \cdot O_p(1) \\
& = O_p\left(N^{-1/2}\right),
\end{aligned}
$$

and the difference between the maximum likelihood estimator and the parameter will also be $O_p\left(N^{-1/2}\right)$, that is,

$$
\left\|\widetilde{\theta}_\tau - \theta_\tau\right\| = O_p\left(N^{-1/2}\right).
$$

Now we find the convergence order of each of the previous terms of the sum, from φ_1 to φ_{10}:

$$
\begin{aligned}
\varphi_1 & = \left|\widehat{c}_{1,\tau} - c_{1,\tau}\right| \cdot \left\|\theta_\tau\right\| \cdot \left|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^{\intercal}\theta_\tau\right)\right| \\
& = O_p\left(N^{(a-1)/2}\right) \cdot O_p(1) \cdot O_p(1) = O_p\left(N^{(a-1)/2}\right),
\end{aligned}
$$

$$\begin{aligned}\varphi_2 &= \left|\widehat{c}_{1,\tau} - c_{1,\tau}\right| \cdot \|\theta_\tau\| \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal \right\| \\ &= O_p\left(N^{(a-1)/2}\right) \cdot O_p\left(1\right) \cdot O_p\left(N^{-1/2}\right) \cdot O_p\left(1\right) = O_p\left(N^{a/2-1}\right),\end{aligned}$$

$$\begin{aligned}\varphi_3 &= \left|\widehat{c}_{1,\tau} - c_{1,\tau}\right| \cdot \left\|\widetilde{\theta}_\tau - \theta_\tau\right\| \cdot \left| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right) \right| \\ &= O_p\left(N^{(a-1)/2}\right) \cdot O_p\left(N^{-1/2}\right) \cdot O_p\left(1\right) = O_p\left(N^{a/2-1}\right),\end{aligned}$$

$$\begin{aligned}\varphi_4 &= \left|\widehat{c}_{1,\tau} - c_{1,\tau}\right| \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\| \cdot \left\|\widetilde{\theta}_\tau - \theta_\tau\right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal \right\| \\ &= O_p\left(N^{(a-1)/2}\right) \cdot O_p\left(N^{-1/2}\right) \cdot O_p\left(N^{-1/2}\right) \cdot O_p\left(1\right) = O_p\left(N^{(a-3)/2}\right),\end{aligned}$$

$$\begin{aligned}\varphi_5 &= \left|\widehat{c}_{1,\tau} - c_{1,\tau}\right| \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\| \cdot \left| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right) \right| \\ &= O_p\left(N^{(a-1)/2}\right) \cdot O_p\left(N^{-1/2}\right) \cdot O_p\left(1\right) = O_p\left(N^{a/2-1}\right),\end{aligned}$$

$$\begin{aligned}\varphi_6 &= \left|\widehat{c}_{1,\tau} - c_{1,\tau}\right| \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\|^2 \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal \right\| \\ &= O_p\left(N^{(a-1)/2}\right) \cdot O_p\left(N^{-1}\right) \cdot O_p\left(1\right) = O_p\left(N^{(a-3)/2}\right),\end{aligned}$$

$$\begin{aligned}\varphi_7 &= c_{1,\tau} \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\| \cdot \left| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right) \right| \\ &= O_p\left(1\right) \cdot O_p\left(N^{-1/2}\right) \cdot O_p\left(1\right) = O_p\left(N^{-1/2}\right),\end{aligned}$$

$$\begin{aligned}\varphi_8 &= c_{1,\tau} \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\|^2 \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal \right\| \\ &= O_p\left(1\right) \cdot O_p\left(N^{-1}\right) \cdot O_p\left(1\right) = O_p\left(N^{-1}\right),\end{aligned}$$

$$
\begin{aligned}
\varphi_9 &= c_{1,\tau} \cdot \|\theta_\tau\| \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\| \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal\right\| \\
&= O_p(1) \cdot O_p(1) \cdot O_p\left(N^{-1/2}\right) \cdot O_p(1) = O_p\left(N^{-1/2}\right),
\end{aligned}
$$

$$
\begin{aligned}
\varphi_{10} &= c_{1,\tau} \cdot \left\|\widehat{\theta}_\tau - \widetilde{\theta}_\tau\right\| \cdot \left\|\widetilde{\theta}_\tau - \theta_\tau\right\| \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^*\right) \cdot X_i^\intercal\right\| \\
&= O_p\left(N^{-1/2}\right) \cdot O_p\left(N^{-1/2}\right) \cdot O_p(1) = O_p\left(N^{-1}\right).
\end{aligned}
$$

Thus, the order of convergence of $\left|\widehat{UQPE}_{\mathrm{RIF}-logit} - \widetilde{UQPE}_{\mathrm{RIF}-logit}(\tau)\right|$ is $O_p\left(N^{(a-1)/2}\right)$. We now find the order of convergence of the next part of the sum:

$$
\begin{aligned}
&\left|\widetilde{UQPE}_{\mathrm{RIF}-logit}(\tau) - UQPE(\tau)\right| \\
= &\left|c_{1,\tau} \cdot \widetilde{\theta}_\tau \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \widetilde{\theta}_\tau\right) - c_{1,\tau} \cdot \theta_\tau \cdot E\left[\Lambda'\left(X^\intercal \theta_\tau\right)\right]\right| \\
= \; & c_{1,\tau} \cdot \left|\left(\widetilde{\theta}_\tau - \theta_\tau\right) \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \widetilde{\theta}_\tau\right) + \theta_\tau \cdot \left(\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \widetilde{\theta}_\tau\right) - E\left[\Lambda'\left(X^\intercal \theta_\tau\right)\right]\right)\right| \\
\leq \; & c_{1,\tau} \cdot \left\|\widetilde{\theta}_\tau - \theta_\tau\right\| \cdot \left|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right)\right| + c_{1,\tau} \cdot \left\|\widetilde{\theta}_\tau - \theta_\tau\right\|^2 \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^{**}\right) \cdot X_i^\intercal\right\| \\
& + c_{1,\tau} \cdot \|\theta_\tau\| \cdot \left(\frac{1}{N} \cdot \sum_{i=1}^{N} \left(\Lambda'\left(X_i^\intercal \theta_\tau\right) - E\left[\Lambda'\left(X^\intercal \theta_\tau\right)\right]\right)\right) \\
& + c_{1,\tau} \cdot \|\theta_\tau\| \cdot \left\|\widetilde{\theta}_\tau - \theta_\tau\right\| \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda''\left(X_i^\intercal \theta_\tau^{**}\right) \cdot X_i^\intercal\right\| \\
= \; & O_p(1) \cdot O_p\left(N^{-1/2}\right) \cdot O_p(1) + O_p(1) \cdot O_p\left(N^{-1}\right) \cdot O_p(1) \\
& + O_p(1) \cdot O_p(1) \cdot O_p\left(N^{-1/2}\right) + O_p(1) \cdot O_p(1) \cdot O_p\left(N^{-1/2}\right) \cdot O_p(1) \\
= \; & O_p\left(N^{-1/2}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left\|\widehat{UQPE}_{\mathrm{RIF}-logit} - UQPE(\tau)\right\| &= \left\|\left(\widehat{c}_{1,\tau} - c_{1,\tau}\right) \cdot \theta_\tau \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right)\right\| + o_p\left(N^{(a-1)/2}\right) \\
&= O_p\left(N^{(a-1)/2}\right).
\end{aligned}
$$

1.3.2 Asymptotic Normality

$$
\begin{aligned}
& \sqrt{N \cdot b_Y} \cdot \left(\widehat{UQPE}_{\text{RIF}-logit} - UQPE\left(\tau\right) \right) \\
= \; & \sqrt{N \cdot b_Y} \cdot \left(\widehat{c}_{1,\tau} - c_{1,\tau}\right) \cdot \theta_\tau \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right) + o_p\left(1\right) \\
= \; & \sqrt{N \cdot b_Y} \cdot \left(\widehat{c}_{1,\tau} - c_{1,\tau}\right) \cdot \theta_\tau \cdot E\left[\Lambda'\left(X^\intercal \theta_\tau\right)\right] \\
& + \sqrt{N \cdot b_Y} \cdot \left(\widehat{c}_{1,\tau} - c_{1,\tau}\right) \cdot \theta_\tau \cdot \left(\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(X_i^\intercal \theta_\tau\right) - E\left[\Lambda'\left(X^\intercal \theta_\tau\right)\right] \right) + o_p\left(1\right) \\
= \; & \sqrt{N \cdot b_Y} \cdot \left(\widehat{c}_{1,\tau} - c_{1,\tau}\right) \cdot \theta_\tau \cdot E\left[\Lambda'\left(X^\intercal \theta_\tau\right)\right] + o_p\left(1\right) \\
= \; & -\frac{\theta_\tau \cdot E\left[\Lambda'\left(X^\intercal \theta_\tau\right)\right]}{f_Y^2\left(q_\tau\right)} \cdot \sqrt{N \cdot b_Y} \cdot \left(\widehat{f}_Y\left(q_\tau\right) - f_Y\left(q_\tau\right) \right) + o_p\left(1\right) \\
& \xrightarrow{D} N\left(0, f_Y^{-3}\left(q_\tau\right) \cdot E\left[\Lambda'\left(X^\intercal \theta_\tau\right)\right]^2 \cdot \theta_\tau \cdot \theta_\tau^\intercal \cdot \int K_Y^2\left(z\right) \cdot dz \right).
\end{aligned}
$$

■

1.4 RIF $- NP$: asymptotics

PROPOSITION 3 [RIF − ***NP***] *Under the assumptions for the Sieve-series approximation in Hirano, Imbens and Ridder (2003, Assumption 5) and assumptions **1 and, 2***

$$
\begin{aligned}
& \widehat{UQPE}_{\text{RIF}-NP}\left(\tau\right) - UQPE\left(\tau\right) \\
= \; & \frac{1}{N} \cdot \sum_{i=1}^{N} \left(H_{K(\tau)}(X_i)^\intercal \cdot \left(\widehat{\rho}_K\left(\tau\right) - \widetilde{\rho}_K\left(\tau\right)\right) \right) \cdot \left(H'_{K(\tau)}(X_i)^\intercal \cdot \rho_K\left(\tau\right) \right) \cdot \Lambda''\left(H_{K(\tau)}(X_i)^\intercal \rho_K^{**}\left(\tau\right) \right) \\
& + o_p\left(N^{3c-1/2}\right).
\end{aligned}
$$

Proof of Proposition 3:

1.4.1 Order of convergence

We estimate $UQPE\left(\tau\right)$ nonparametrically by means of $\widehat{UQPE}_{\text{RIF}-NP}\left(\tau\right)$:

$$
\widehat{UQPE}_{\text{RIF}-NP}\left(\tau\right) = \widehat{c}_{1,\tau} \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} \frac{d \Pr\left[\widehat{T_\tau = 1 | X} = X_i\right]}{dx},
$$

where the nonparametric estimate of $d\Pr\left[T_\tau = 1|X = x\right]/dx$ is

$$\frac{d\widehat{\Pr\left[T_\tau = 1|X = x\right]}}{dx} = \widehat{\rho}_K\left(\tau\right) \cdot H'_{K(\tau)}(x)^\intercal \cdot \Lambda'\left(H_{K(\tau)}(x)^\intercal \widehat{\rho}_K\left(\tau\right)\right).$$

An unfeasible nonparametric estimator of $UQPE\left(\tau\right)$ is $\widetilde{UQPE}_{\mathrm{RIF}-NP}\left(\tau\right)$:

$$\widetilde{UQPE}_{\mathrm{RIF}-NP}\left(\tau\right) = c_{1,\tau} \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} \frac{d\widetilde{\Pr\left[T_\tau = 1|X = X_i\right]}}{dx},$$

$$\frac{d\widetilde{\Pr\left[T_\tau = 1|X = x\right]}}{dx} = \widetilde{\rho}_K\left(\tau\right) \cdot \frac{dH_{K(\tau)}(x)^\intercal}{dx} \cdot \Lambda'\left(H_{K(\tau)}(x)^\intercal \widetilde{\rho}_K\left(\tau\right)\right),$$

where

$$\widetilde{\rho}_K\left(\tau\right) = \arg\max_{\rho_K(\tau)} \sum_{i}^{N} T_{\tau,i} \cdot H_{K(\tau)}(X_i)^\intercal \rho_K\left(\tau\right) + \log\left(1 - \Lambda\left(H_{K(\tau)}(X_i)^\intercal \rho_K\left(\tau\right)\right)\right).$$

We now establish the order of convergence of $\widehat{UQPE}_{\mathrm{RIF}-NP}\left(\tau\right) - \widetilde{UQPE}_{\mathrm{RIF}-NP}\left(\tau\right)$:

$$\begin{aligned}
&\left\|\widehat{UQPE}_{\mathrm{RIF}-NP}\left(\tau\right) - \widetilde{UQPE}_{\mathrm{RIF}-NP}\left(\tau\right)\right\| \\
\leq\ & \left|\widehat{c}_{1,\tau} - c_{1,\tau}\right| \cdot \left\|\frac{UQPE\left(\tau\right)}{c_{1,\tau}}\right\| \\
&+ \left|\widehat{c}_{1,\tau} - c_{1,\tau}\right| \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N}\left(\frac{d\widetilde{\Pr\left[T_\tau = 1|X = X_i\right]}}{dx} - \frac{UQPE\left(\tau\right)}{c_{1,\tau}}\right)\right\| \\
&+ c_{1,\tau} \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N}\left(\frac{d\widehat{\Pr\left[T_\tau = 1|X = x\right]}}{dx} - \frac{d\widetilde{\Pr\left[T_\tau = 1|X = X_i\right]}}{dx}\right)\right\| \\
&+ \left|\widehat{c}_{1,\tau} - c_{1,\tau}\right| \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N}\left(\frac{d\widehat{\Pr\left[T_\tau = 1|X = x\right]}}{dx} - \frac{d\widetilde{\Pr\left[T_\tau = 1|X = X_i\right]}}{dx}\right)\right\|.
\end{aligned}$$

Note that

$$\left\|\frac{1}{N} \cdot \sum_{i=1}^{N}\left(\frac{d\widetilde{\Pr\left[T_\tau = 1|X = X_i\right]}}{dx} - \frac{UQPE\left(\tau\right)}{c_{1,\tau}}\right)\right\| = O_p\left(N^{-1/2}\right),$$

as the unfeasible estimator will simply be an average derivative estimator whose properties have been already established in the literature (Härdle and Stoker, 1989). The key is to derive the order of convergence of

$$\left\| \frac{1}{N} \cdot \sum_{i=1}^{N} \left(\frac{d \widehat{\Pr\left[T_\tau = 1 | X = x\right]}}{dx} - \frac{d \widetilde{\Pr\left[T_\tau = 1 | X = X_i\right]}}{dx} \right) \right\|$$

$$\leq \left\| \widehat{\rho}_K\left(\tau\right) - \widetilde{\rho}_K\left(\tau\right) \right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} H'_{K(\tau)}(X_i)^\intercal \cdot \Lambda'\left(H_{K(\tau)}(X_i)^\intercal \widetilde{\rho}_K\left(\tau\right)\right) \right\|$$

$$+ \left\| \widehat{\rho}_K\left(\tau\right) \right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} H'_{K(\tau)}(X_i)^\intercal \cdot \left(\Lambda'\left(H_{K(\tau)}(X_i)^\intercal \widehat{\rho}_K\left(\tau\right)\right) - \Lambda'\left(H_{K(\tau)}(X_i)^\intercal \widetilde{\rho}_K\left(\tau\right)\right) \right) \right\|$$

$$\leq \left\| \widehat{\rho}_K\left(\tau\right) - \widetilde{\rho}_K\left(\tau\right) \right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} H'_{K(\tau)}(X_i)^\intercal \cdot \Lambda'\left(H_{K(\tau)}(X_i)^\intercal \rho_K\left(\tau\right)\right) \right\|$$

$$+ \left\| \widehat{\rho}_K\left(\tau\right) - \widetilde{\rho}_K\left(\tau\right) \right\| \cdot \left\| \widetilde{\rho}_K\left(\tau\right) - \rho_K\left(\tau\right) \right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} H'_{K(\tau)}(X_i)^\intercal \cdot H_{K(\tau)}(X_i) \cdot \Lambda''\left(H_{K(\tau)}(X_i)^\intercal \rho^{*}_K\left(\tau\right)\right) \right\|$$

$$+ \left\| \rho_K\left(\tau\right) \right\| \cdot \left\| \widehat{\rho}_K\left(\tau\right) - \widetilde{\rho}_K\left(\tau\right) \right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} H'_{K(\tau)}(X_i)^\intercal \cdot H_{K(\tau)}(X_i) \cdot \Lambda''\left(H_{K(\tau)}(X_i)^\intercal \rho^{**}_K\left(\tau\right)\right) \right\|$$

$$+ \left\| \widetilde{\rho}_K\left(\tau\right) - \rho_K\left(\tau\right) \right\| \cdot \left\| \widehat{\rho}_K\left(\tau\right) - \widetilde{\rho}_K\left(\tau\right) \right\| \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} H'_{K(\tau)}(X_i)^\intercal \cdot H_{K(\tau)}(X_i) \cdot \Lambda''\left(H_{K(\tau)}(X_i)^\intercal \rho^{**}_K\left(\tau\right)\right) \right\|$$

$$+ \left\| \widehat{\rho}_K\left(\tau\right) - \widetilde{\rho}_K\left(\tau\right) \right\|^2 \cdot \left\| \frac{1}{N} \cdot \sum_{i=1}^{N} H'_{K(\tau)}(X_i)^\intercal \cdot H_{K(\tau)}(X_i) \cdot \Lambda''\left(H_{K(\tau)}(X_i)^\intercal \rho^{**}_K\left(\tau\right)\right) \right\|,$$

where $\rho_K\left(\tau\right)$ is the pseudo true value $\rho_K\left(\tau\right)$ as

$$\rho_K\left(\tau\right) \equiv \arg\max_{\rho} E\left[\left[\Pr\left[T_\tau = 1 | X\right] \cdot H_{K(\tau)}(X)^\intercal \rho + \log\left(1 - \Lambda\left(H_{K(\tau)}(X)^\intercal \rho\right)\right)\right]\right].$$

Consider the following arguments:

1. Order of $\left\|\left(\widehat{\rho}_K\left(\tau\right) - \widetilde{\rho}_K\left(\tau\right)\right)\right\|$:

$$\begin{aligned} \widehat{\rho}_K\left(\tau\right) &= \arg_{\rho_K} zero \sum_{i=1}^{N} H_{K(\tau)}(X_i)^\intercal \cdot \left(\widehat{T}_{\tau,i} - \Lambda\left(H_{K(\tau)}(X_i)^\intercal \rho_K\left(\tau\right)\right)\right) \\ \widetilde{\rho}_K\left(\tau\right) &= \arg_{\rho_K} zero \sum_{i=1}^{N} H_{K(\tau)}(X_i)^\intercal \cdot \left(T_{\tau,i} - \Lambda\left(H_{K(\tau)}(X_i)^\intercal \rho_K\left(\tau\right)\right)\right). \end{aligned}$$

Thus,

$$
\begin{aligned}
& \widehat{\rho}_K(\tau) - \widetilde{\rho}_K(\tau) \\
= & \left(\sum_{i=1}^{N} H_{K(\tau)}(X_i) \cdot H_{K(\tau)}(X_i)^\intercal \cdot \Lambda' \left(H_{K(\tau)}(X_i)^\intercal \rho_K^*(\tau) \right) \right)^{-1} \\
& \cdot \sum_{i=1}^{N} H_{K(\tau)}(X_i)^\intercal \cdot \left(\widehat{T}_{\tau,i} - T_{\tau,i} \right)
\end{aligned}
$$

Thus, to find the order of convergence of this term, we proceed as follows:

$$
\begin{aligned}
& \left\| \left(\widehat{\rho}_K(\tau) - \widetilde{\rho}_K(\tau) \right) \right\| \\
= & \left\| \left(\frac{1}{N} \cdot \sum_{i=1}^{N} H_{K(\tau)}(X_i) \cdot H_{K(\tau)}(X_i)^\intercal \cdot \Lambda' \left(H_{K(\tau)}(X_i)^\intercal \rho_K^*(\tau) \right) \right)^{-1} \right. \\
& \left. \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} H_{K(\tau)}(X_i)^\intercal \cdot R\left(Y_i, \widehat{q}_\tau, q_\tau \right) \right\| \\
\leq & \; C \cdot \left\| \sup_{x \in \mathcal{X}} H_{K(\tau)}(x) \right\| \cdot \left| \frac{1}{N} \cdot \sum_{i=1}^{N} R\left(Y_i, \widehat{q}_\tau, q_\tau \right) \right| \\
= & \; O_p\left(\Gamma(K) \cdot N^{-1/2} \right) = O_p\left(K(N) \cdot N^{-1/2} \right) \\
= & \; O_p\left(N^{c-1/2} \right),
\end{aligned}
$$

where

$$
\left\| \sup_{x \in \mathcal{X}} H_{K(\tau)}(x) \right\| = \Gamma(K) = O(K),
$$

but the length of the vector of polynomials depends on N at the following way:

$$
K(N) = O_p\left(N^c \right), \qquad c > 0.
$$

2. Order of $\left\| \widetilde{\rho}_K(\tau) - \rho_K(\tau) \right\|$ and of $\left\| \rho_K(\tau) \right\|$:

   We have

$$
\widetilde{\rho}_K(\tau) = \rho_K(\tau) + \widetilde{\rho}_K(\tau) - \rho_K(\tau),
$$

but remember that

$$
\begin{aligned}
0 &= \sum_{i=1}^{N} H_{K(\tau)}(X_i) \cdot \left(T_{\tau,i} - \Lambda\left(H_{K(\tau)}(X_i)^{\intercal} \widetilde{\rho}_K(\tau)\right)\right) \\
&= \sum_{i=1}^{N} H_{K(\tau)}(X_i) \cdot \Big(T_{\tau,i} - \Lambda\left(H_{K(\tau)}(X_i)^{\intercal} \rho_K(\tau)\right) - \Lambda'\left(H_{K(\tau)}(X_i)^{\intercal} \rho_K^{*}(\tau)\right) \\
&\qquad \cdot H_{K(\tau)}(X_i)^{\intercal}\left(\widetilde{\rho}_K(\tau) - \rho_K(\tau)\right)\Big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\widetilde{\rho}_K(\tau) - \rho_K(\tau) &= \left(\frac{1}{N} \cdot \sum_{i=1}^{N} H_{K(\tau)}(X_i) \cdot H_{K(\tau)}(X_i)^{\intercal} \cdot \Lambda'\left(H_{K(\tau)}(X_i)^{\intercal} \rho_K^{*}(\tau)\right)\right)^{-1} \\
&\quad \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} H_{K(\tau)}(X_i)^{\intercal} \cdot \left(T_{\tau,i} - \Lambda\left(H_{K(\tau)}(X_i)^{\intercal} \rho_K(\tau)\right)\right).
\end{aligned}
$$

But

$$
\begin{aligned}
& E\left[\left\| \frac{1}{\sqrt{N}} \cdot \sum_{i=1}^{N} H_{K(\tau)}(X_i)^{\intercal} \cdot \left(T_{\tau,i} - \Lambda\left(H_{K(\tau)}(X_i)^{\intercal} \rho_K(\tau)\right)\right) \right\|^2\right] \\
= \; & tr\left(E\left[H_{K(\tau)}(X) \cdot H_{K(\tau)}(X)^{\intercal} \cdot \Lambda'\left(H_{K(\tau)}(X)^{\intercal} \rho_K(\tau)\right)\right]\right) \\
\leq \; & C \cdot K,
\end{aligned}
$$

as $H_{K(\tau)}(X)$ is a rotation of an orthogonal polynomial. Now under the hypothesis that $K^4/N \to 0$

$$
\left\|\widetilde{\rho}_K(\tau) - \rho_K(\tau)\right\| = O_p\left((K/N)^{1/2}\right).
$$

Now consider the difference between $\rho_K(\tau)$ and $\rho_{0,K}(\tau)$ where is such that, according to Newey (1995, 1997):

$$
\sup_{x \in \mathcal{X}} \left| \ln\left(\Pr\left[T_\tau = 1 | X\right]\right) - \ln\left(1 - \Pr\left[T_\tau = 1 | X\right]\right) - H_{K(\tau)}(X)^{\intercal} \rho_{0,K}(\tau) \right| \leq C \cdot K^{-s/k}
$$

where s is the number of derivatives of $m_\tau(x)$, and k the length of X. Following Hirano, Imbens, and Ridder (2003):

$$
\left\|\rho_K(\tau) - \rho_{0,K}(\tau)\right\| \leq C \cdot K^{-s/(2r)}.
$$

Thus

$$\begin{aligned}
\|\rho_K(\tau)\| &\leq \left\|\rho_K(\tau) - \rho_{0,K}(\tau)\right\| + \left\|\rho_{0,K}(\tau)\right\| \\
&= O\left(K^{-s/(2r)}\right) + \Gamma(K) \\
&= O(K).
\end{aligned}$$

3. Order of $\left\|\frac{1}{N} \cdot \sum_{i=1}^{N} H'_{K(\tau)}(X_i)^\intercal \cdot \Lambda'\left(H_{K(\tau)}(X_i)^\intercal \rho_K(\tau)\right)\right\|$:

$$\begin{aligned}
&\left\|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(H_{K(\tau)}(X_i)^\intercal \rho_K(\tau)\right) \cdot H'_{K(\tau)}(X_i)^\intercal\right\| \\
\leq\ & C \cdot \left\|\sup_{x\in\mathcal{X}} H_{K(\tau)}(x)\right\| \cdot \left\|\frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'\left(H_{K(\tau)}(X_i)^\intercal \rho_K(\tau)\right)\right\| \\
\leq\ & O(K) \cdot O_p(1) = O_P(N^c)
\end{aligned}$$

4. Order of $\left\|\frac{1}{N} \cdot \sum_{i=1}^{N} H'_{K(\tau)}(X_i)^\intercal \cdot H_{K(\tau)}(X_i) \cdot \Lambda''\left(H_{K(\tau)}(X_i)^\intercal \rho_K^{**}(\tau)\right)\right\|$:

   The derivative vector $H'_{K(\tau)}(x)$ is just a linear combination of the original polynomial vector
$$H'_{K(\tau)}(x) = A \cdot H_{K(\tau)}(x).$$

   Therefore

$$\begin{aligned}
&\left\|\frac{1}{N} \cdot \sum_{i=1}^{N} H'_{K(\tau)}(X_i)^\intercal \cdot H_{K(\tau)}(X_i) \cdot \Lambda''\left(H_{K(\tau)}(X_i)^\intercal \rho_K^{**}(\tau)\right)\right\| \\
\leq\ & C \cdot tr\left(E\left[H_{K(\tau)}(X) \cdot H_{K(\tau)}(X)^\intercal \cdot \Lambda''\left(H_{K(\tau)}(X)^\intercal \rho_K(\tau)\right)\right]\right) \\
&+C \cdot \|\xi\| \\
=\ & O(K) + O_p(\|\xi\|) = O_p(K(N)) + o_p(K(N)) \\
=\ & O_p(N^c),
\end{aligned}$$

   where

$$\begin{aligned}
\xi &= \frac{1}{N} \cdot \sum_{i=1}^{N} H_{K(\tau)}(X_i)^\intercal \cdot H_{K(\tau)}(X_i) \cdot \Lambda''\left(H_{K(\tau)}(X_i)^\intercal \rho_K^{**}(\tau)\right) \\
&\quad -E\left[H_{K(\tau)}(X) \cdot H_{K(\tau)}(X)^\intercal \cdot \Lambda''\left(H_{K(\tau)}(X)^\intercal \rho_K(\tau)\right)\right].
\end{aligned}$$

Finally

$$\|\widehat{\rho}_K(\tau)-\widetilde{\rho}_K(\tau)\|\cdot\left\|\frac{1}{N}\cdot\sum_{i=1}^{N}H'_{K(\tau)}(X_i)^{\intercal}\cdot\Lambda'\left(H_{K(\tau)}(X_i)^{\intercal}\rho_K(\tau)\right)\right\|$$
$$=O_p\left(N^{c-1/2}\right)\cdot O_p\left(N^{c}\right)=O_p\left(N^{2c-1/2}\right),$$

$$\|\widehat{\rho}_K(\tau)-\widetilde{\rho}_K(\tau)\|$$
$$\cdot\|\widetilde{\rho}_K(\tau)-\rho_K(\tau)\|\cdot\left\|\frac{1}{N}\cdot\sum_{i=1}^{N}H'_{K(\tau)}(X_i)^{\intercal}\cdot H_{K(\tau)}(X_i)\cdot\Lambda''\left(H_{K(\tau)}(X_i)^{\intercal}\rho^{*}_K(\tau)\right)\right\|$$
$$=O_p\left(N^{c-1/2}\right)\cdot O_p\left(N^{(c-1)/2}\right)\cdot O_p\left(N^{c}\right)=O_p\left(N^{5c-1}\right),$$

$$\|\rho_K(\tau)\|\cdot\|\widehat{\rho}_K(\tau)-\widetilde{\rho}_K(\tau)\|\cdot\left\|\frac{1}{N}\cdot\sum_{i=1}^{N}H'_{K(\tau)}(X_i)^{\intercal}\cdot H_{K(\tau)}(X_i)\cdot\Lambda''\left(H_{K(\tau)}(X_i)^{\intercal}\rho^{**}_K(\tau)\right)\right\|$$
$$=O\left(N^{c}\right)\cdot O_p\left(N^{c-1/2}\right)\cdot O_p\left(N^{c}\right)=O_p\left(N^{3c-1/2}\right),$$

$$\|\widetilde{\rho}_K(\tau)-\rho_K(\tau)\|\cdot\|\widehat{\rho}_K(\tau)-\widetilde{\rho}_K(\tau)\|$$
$$\cdot\left\|\frac{1}{N}\cdot\sum_{i=1}^{N}H'_{K(\tau)}(X_i)^{\intercal}\cdot H_{K(\tau)}(X_i)\cdot\Lambda''\left(H_{K(\tau)}(X_i)^{\intercal}\rho^{**}_K(\tau)\right)\right\|$$
$$=\ O_p\left(N^{(c-1)/2}\right)\cdot O_p\left(N^{c-1/2}\right)\cdot O_p\left(N^{c}\right)=O_p\left(N^{5c-1}\right),$$

$$\|\widehat{\rho}_K(\tau)-\widetilde{\rho}_K(\tau)\|^2\cdot\left\|\frac{1}{N}\cdot\sum_{i=1}^{N}H'_{K(\tau)}(X_i)^{\intercal}\cdot H_{K(\tau)}(X_i)\cdot\Lambda''\left(H_{K(\tau)}(X_i)^{\intercal}\rho^{**}_K(\tau)\right)\right\|$$
$$=\ O_p\left(N^{2c-1}\right)\cdot O_p\left(N^{c}\right)=O_p\left(N^{3c-1}\right).$$

Thus

$$\begin{aligned}&\left\|\frac{1}{N}\cdot\sum_{i=1}^{N}\left(\frac{d\Pr\widehat{[T_\tau=1|}X=x]}{dx}-\frac{d\Pr\widetilde{[T_\tau=1|}X=X_i]}{dx}\right)\right\|\\=\ &\|\rho_K(\tau)\|\cdot\|\widehat{\rho}_K(\tau)-\widetilde{\rho}_K(\tau)\|\\&\cdot\left\|\frac{1}{N}\cdot\sum_{i=1}^{N}H'_{K(\tau)}(X_i)^{\intercal}\cdot H_{K(\tau)}(X_i)\cdot\Lambda''\left(H_{K(\tau)}(X_i)^{\intercal}\rho^{**}_K(\tau)\right)\right\|+o_p\left(N^{3c-1/2}\right)\\=\ &O_p\left(N^{3c-1/2}\right).\end{aligned}$$

and, therefore

$$
\begin{aligned}
&\widehat{UQPE}_{\mathrm{RIF}-NP}(\tau) - UQPE(\tau) \\
&= \frac{1}{N} \cdot \sum_{i=1}^{N} \left(H_{K(\tau)}(X_i)^{\intercal} \cdot \left(\widehat{\rho}_K(\tau) - \widetilde{\rho}_K(\tau) \right) \right) \cdot \left(H'_{K(\tau)}(X_i)^{\intercal} \cdot \rho_K(\tau) \right) \cdot \Lambda'' \left(H_{K(\tau)}(X_i)^{\intercal} \rho^{**}_K(\tau) \right) \\
&\qquad + o_p\left(N^{3c-1/2} \right).
\end{aligned}
$$

■

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